Lecture 2

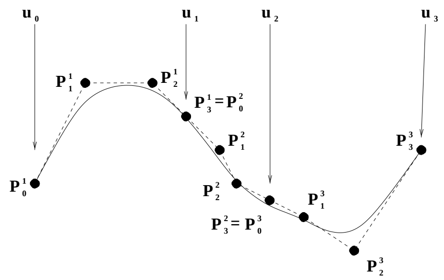
NURBS Surfaces

Piecewise Curves

Curves consisting of just one polynomial segment are often insufficient:

- A high degree is required in order to satisfy a large number of constraints (data points) - inefficient & numerically unstable.
- Single-segment curves are not well-suited to interactive shape design.
- The control of Bézier curves is not sufficiently local.

Solution: use a curve that is constructed piecewise:



The curve consists of m(=3) n^{th}-degree segments. Each segment is denoted by $\mathbf{C}_i(u)$, $1 \leq i \leq m$. The $u_0 = 0 < u_1 < u_2 < u_3 = 1$ are called (breakpoints) knots. In this case a Bézier curve has been used.

Continuity

Segments are constructed so, at the join, they have some level of continuity. $\mathbf{C}_i^j$ is the j^{th} derivative of $\mathbf{C}_i$. If the curve has the same tangents at the join (i.e. both slope and magnitude) the curve is said to have first-degree continuity $\mathbf{C}^1$. For camera animations, it is preferable to enforce $\mathbf{C}^2$ continuity.

If, in the previous figure, $n = 3$ and the knots $U = \{u_0, u_1, u_2, u_3\}$ remain fixed and the $\mathbf{P}_i^j$ varied arbitrarily, then there are 12 free parameters. Forcing the end & start points of each segment to be in the same position (e.g. $\mathbf{P}_3^1 = \mathbf{P}_0^2$) forces $\mathbf{C}^0$ continuity, and there are now 10 free parameters. Enforcing $\mathbf{C}^1$ continuity is slightly more complex (points around each end point lie on a straight line) and leads to only 8 free parameters.

It is clear that manipulating individual polynomial segments is not ideal since:

- Redundant data must be stored - 12 coefficients where only 8 are required for $\mathbf{C}^1$ cubic curves.
- If we are happy with curve segments $\mathbf{C}_1$ and $\mathbf{C}_3$ and desire $\mathbf{C}^1$ continuity, then we **cannot** alter $\mathbf{C}_2$.

NonRational B-Splines

We require that:

- Continuity is determined by the basis functions, not the control points.
- 'Nice' properties (e.g. convex hull, transformation invariance) are maintained.
- The basis functions should have local support i.e. each is nonzero only on a limited number of subintervals, not the entire domain $[u_0, \ldots, u_m]$.

A p^{th}-degree B-spline curve :

$$\mathbf{C}(u) = \sum_{i=0}^{n} N_{i,p}(u)\mathbf{P}_i \qquad a \leq u \leq b$$

where the $\{\mathbf{P}_i\}$ are control points and the $\{N_{i,p}\}$ are the p^{th}-degree B-spline basis functions defined on a nonuniform knot vector:

$$U = \{a, \ldots, a, u_{p+1}, \ldots, u_{m-p-1}, b, \ldots, b\}$$

Nonuniform means that the knots need not be regularly spaced. In this course we shall only consider the case of **clamped** knot vectors, where the first and last $(p+1)$ components of the knot vector are a and b. Typically this means $U = \{0, 0, 0, 0, \ldots, 1, 1, 1, 1\}$.

B-Spline Basis Functions

The basis functions are defined recursively:

$$N_{i,0}(u) = \begin{cases} 1 & \text{if } u_i \le u < u_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$N_{i,p}(u) = \frac{u - u_i}{u_{i+p} - u_i} N_{i,p-1}(u) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u)$$

Note that:

- $N_{i,0}(u)$ is a step function, equal to zero everywhere except on the interval $u_i \le u < u_{i+1}$. This interval is know as the i^{th} knot span. It can have zero length, since knots need not be distinct.

- The equations can yield $\frac{0}{0}$ - this is defined here as zero.

An Example B-Spline Basis

For $U = \{0, 0, 0, 1, 1, 1\}$, $p = 2$ and considering only u in the range $[0, 1)$:

$$\begin{aligned}
N_{0,0} &= 0 \\
N_{1,0} &= 0 \\
N_{2,0} &= 1 \\
N_{3,0} &= 0 \\
N_{0,4} &= 0
\end{aligned}$$

$$\begin{aligned}
N_{0,1} &= \tfrac{u-0}{0-0} N_{0,0} + \tfrac{0-u}{0-0} N_{1,0} &&= 0 \\
N_{1,1} &= \tfrac{u-0}{0-0} N_{1,0} + \tfrac{1-u}{1-0} N_{2,0} &&= 1 - u \\
N_{2,1} &= \tfrac{u-0}{1-0} N_{2,0} + \tfrac{1-u}{1-1} N_{3,0} &&= u \\
N_{3,1} &= \tfrac{u-1}{1-1} N_{3,0} + \tfrac{1-u}{1-1} N_{4,0} &&= 0
\end{aligned}$$

$$\begin{aligned}
N_{0,2} &= \tfrac{u-0}{0-0} N_{0,1} + \tfrac{1-u}{1-0} N_{1,1} &&= (1-u)^2 \\
N_{1,2} &= \tfrac{u-0}{1-0} N_{1,1} + \tfrac{1-u}{1-1} N_{2,1} &&= 2u(1-u) \\
N_{2,2} &= \tfrac{u-1}{1-1} N_{2,1} + \tfrac{1-u}{1-1} N_{3,1} &&= u^2
\end{aligned}$$

These are the Bernstein polynomials, as used in Bézier curves. B-splines may be thought of as a generalisation of the Bézier representation if:

$$U = \{\underbrace{0, \ldots, 0}_{p+1}, \underbrace{1, \ldots, 1}_{p+1}\}$$

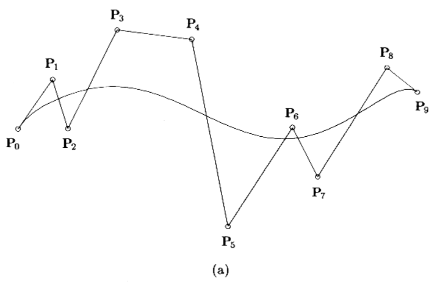
Changing Degree

A degree-9 curve on the knot vector:

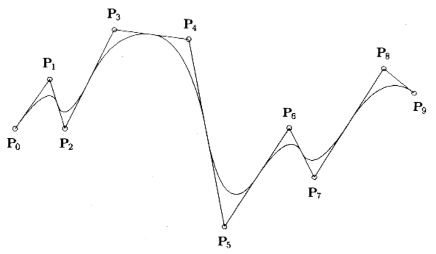
$$U = \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}$$

and a degree-2 curve on:

$$U = \{0, 0, 0, \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8}, 1, 1, 1\}$$



(a)

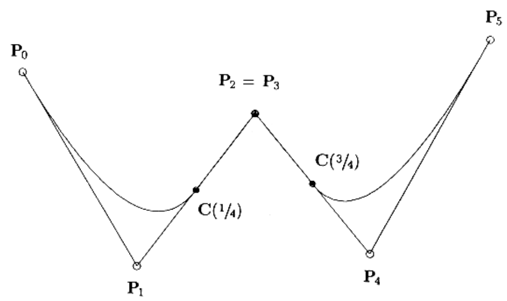


(b)

Multiple Control Points

A quadratic curve where:

$$U = \{0, 0, 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1, 1, 1\}$$

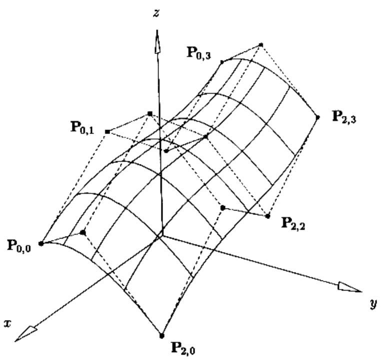


NonRational Bézier Surfaces

A surface is defined as a n x m net of control points, which has both u and v dimension.

$$S(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} B_{i,n}(u)B_{j,m}(v)\mathbf{P}_{i,j} \qquad 0 \le u,v \le 1$$

The Bernstein polynomials make another appearance. Below is shown a 3x5 net, a quadratic x cubic Bézier surface:



NURBS Surfaces

A NURBS surface of degree p in the u direction and degree q in the v direction has the form:

$$S(u,v) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{m} N_{i,p}(u)N_{j,q}(v)w_{i,j}\mathbf{P}_{i,j}}{\sum_{i=0}^{n}\sum_{j=0}^{m} N_{i,p}(u)N_{j,q}(v)w_{i,j}}$$

The $\{\mathbf{P}_{i,j}\}$ from a bidirectional control net, the $\{w_{i,j}\}$ are the weights, and the $\{N_{i,p}(u)\}$ and $\{N_{j,q}(u)\}$ are the nonrational B-spline basis functions defined on the knot vectors:

$$U = \{\underbrace{0,\ldots,0}_{p+1}, u_{p+1},\ldots,u_{r-p-1},\underbrace{1,\ldots,1}_{p+1}\}$$

$$V = \{\underbrace{0,\ldots,0}_{q+1}, u_{q+1},\ldots,u_{s-q-1},\underbrace{1,\ldots,1}_{q+1}\}$$

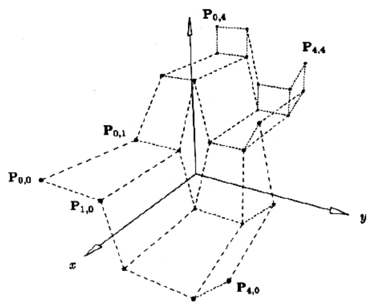
where $r = n + p + 1$ and $s = m + q + 1$.

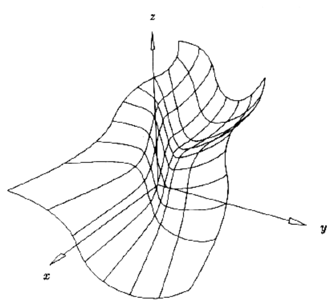
NURBS Example

A biquadratic NURBS surface where $w_{1,1}$ = $w_{1,2}$ = $w_{2,1}$ = $w_{2,2}$ = 10, all other weights are unity.

$$U = V = \{0,0,0,\frac{1}{3},\frac{2}{3},1,1,1\}$$

i.e. a uniform knot vector.

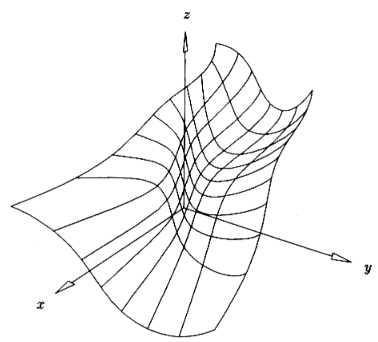




NURBS Example

Now a bicubic surface, with the weights as before, but:

$$U = V = \{0,0,0,0,\frac{1}{2},1,1,1,1\}$$



NURBS

"Nobody understands rational B-splines"

NU	Non-Uniform	Knot Vectors with (possibly) uneven spans.
R	Rational	Use of weights
BS	B-splines	Basis functions, piecewise local curves.

Extremely powerful and complex. The use of multiple knots, repeated control points and rational weights all add to the complexity.

Properties:

- Corner point interpolation e.g. $S(0,0) = \mathbf{P}_{0,0}$.
- Affine invariance.
- Convex Hull
- Local modification; if $\mathbf{P}_{i,j}$ is moved, or $w_{i,j}$ altered, it affects the surface shape only in the rectangle $[u_i, u_{i+p+1}) \times [v_j, v_{j+q+1})$.

For most surfaces in modelling tools Uniform, NonRational B-splines are sufficient.