POSSIBILITY MEASURES, RANDOM SETS AND NATURAL EXTENSION

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ABSTRACT. We study the relationship between possibility and necessity measures defined on arbitrary spaces, the theory of imprecise probabilities, and elementary random set theory. We show how special random sets can be used to generate normal possibility and necessity measures, as well as their natural extensions. This leads to interesting alternative formulas for the calculation of these natural extensions.

1. INTRODUCTION

Possibility measures were introduced by Zadeh [21] in 1978. In his view, these supremum preserving set functions are a mathematical representation of the information conveyed by typical affirmative statements in natural language. For recent discussions of this interpretation, we refer to [7, 20].

Supremum preserving set functions can also be found in the literature under a number of different guises. For instance, they appear in Shackle's logic of surprise [16], and were studied in a measure-theoretic context by Shilkret [18]. They also play the part of special limiting cases in Shafer's theory of belief functions [17]. In a forthcoming paper, we study in detail how possibility measures fit into Walley's behavioural theory of imprecise probabilities [19]. Preliminary results in this field have been published in a conference paper [8]. Also relevant is the related work by Dubois and Prade [12, 13]. In this paper, we discuss the relationship between possibility measures in the context of imprecise probabilities on the one hand, and random sets on the other hand.

We give a short overview of the relevant basic definitions. We work with a nonempty set Ω, called the *universe of discourse*. A *possibility measure* Π on $(\Omega, \wp(\Omega))$ is a mapping from the power class $\wp(\Omega)$ of Ω to the real unit interval $[0,1]$, which is supremum preserving in the following sense: for any family $(A_j \mid j \in J)$ of subsets of Ω,

$$\Pi(\bigcup_{j\in J} A_j) = \sup_{j\in J} \Pi(A_j).$$

Such a possibility measure is completely determined by its *distribution* $\pi\colon \Omega \to [0,1]$, defined by $\pi(\omega) = \Pi(\{\omega\})$, $\omega \in \Omega$. Indeed, for any $A \in \wp(\Omega)$, $\Pi(A) = \sup_{\omega\in A} \pi(\omega)$. Note that, by definition, $\Pi(\emptyset) = 0$. Π is called *normal* iff $\Pi(\Omega) = 1$.

With Π we may associate a dual *necessity measure* $\mathrm{N}\colon \wp(\Omega) \to [0,1]$, defined by $\mathrm{N}(A) = 1 - \Pi(\mathrm{co}A)$, $A \subseteq \Omega$, where $\mathrm{co}A$ denotes the set-theoretic complement of A (relative to Ω). N is infimum preserving, and completely determined by its *distribution* $\nu\colon \Omega \to [0,1]$, defined by $\nu(\omega) = \mathrm{N}(\mathrm{co}\{\omega\}) = 1 - \pi(\omega)$, $\omega \in \Omega$. For any $A \in \wp(\Omega)$, $\mathrm{N}(A) = \inf_{\omega\in \mathrm{co}A} \nu(\omega)$. By definition, $\mathrm{N}(\Omega) = 1$, and we call N *normal* iff Π is, i.e. iff $\mathrm{N}(\emptyset) = 0$. For more details about the theory of possibility measures, we refer to [4, 5, 6, 9, 11, 21].

As mentioned above, possibility and necessity measures can be incorporated into the behavioural theory of imprecise probabilities. Let us briefly describe how this is done. We limit ourselves here to definitions and results which are relevant in the context of this paper. For a detailed account of the theory of imprecise probabilities, we refer to the book by Walley [19].

The universe of discourse Ω can be interpreted as a possibility space, that is, the set of the mutually exclusive possible outcomes of a specific experiment. A *gamble* X on Ω is a bounded real-valued function on Ω, and can be interpreted as an uncertain reward. The set of all gambles on Ω is denoted by $\mathcal{L}(\Omega)$. An *event* A in Ω is a subset of Ω. The set of all events in Ω has already been given the notation $\wp(\Omega)$. We identify events with their characteristic functions, and interpret them as $0-1$-valued gambles. We also denote a *constant gamble* on Ω by the unique real value it assumes. The pointwise order on $\mathcal{L}(\Omega)$ is denoted by $\leq$, i.e. $X \leq Y$ iff $(\forall\omega \in \Omega)(X(\omega) \leq Y(\omega))$.

An *upper prevision* $\overline{P}$ is a real-valued function on a set of gambles $\mathcal{G} \subseteq \mathcal{L}(\Omega)$. In order to identify its domain and possibility space, it is often denoted as $(\Omega, \mathcal{G}, \overline{P})$. The corresponding *lower prevision* $(\Omega, -\mathcal{G}, \underline{P})$ is defined on the domain $-\mathcal{G} = \{-X \mid X \in \mathcal{G}\}$ as $\underline{P}(X) = -\overline{P}(-X)$, $X \in -\mathcal{G}$. In the behavioural context, $\overline{P}(X)$ can be interpreted as an infimum price for selling the gamble X, and $\underline{P}(X)$ as a supremum price for buying it. If $\mathcal{G}$ is in particular a class of events, then $\overline{P}$ is called an *upper probability*. The corresponding *lower probability* $\underline{P}$ is then defined on the set $\{\mathrm{co}A \mid A \in \mathcal{G}\}$ by $\underline{P}(A) = 1 - \overline{P}(\mathrm{co}A)$.

The set $\mathcal{L}(\Omega)$ is a linear space when provided with the pointwise addition of gambles and the pointwise scalar multiplication of gambles with real numbers. A linear functional P on $\mathcal{L}(\Omega)$ which is positive ($X \geq 0 \Rightarrow P(X) \geq 0$) and has unit norm ($P(1) = 1$) is called a *linear prevision* on $\mathcal{L}(\Omega)$. Its restriction to $\wp(\Omega)$ is called a (finitely) *additive probability* on $\wp(\Omega)$. Note that $P(-X) = -P(X)$, $X \in \mathcal{L}(\Omega)$, which means that as an upper prevision, P is equal to the corresponding lower prevision. The set of linear previsions on $\mathcal{L}(\Omega)$ is denoted by $\mathcal{P}(\Omega)$. If $\mathcal{G}$ is a subset of $\mathcal{L}(\Omega)$, a functional on $\mathcal{G}$ is called a linear prevision on $\mathcal{G}$ iff it is the restriction to $\mathcal{G}$ of a linear prevision on $\mathcal{L}(\Omega)$. A similar definition is given for additive probabilities on arbitrary classes of events.

Given an upper prevision $(\Omega, \mathcal{G}, \overline{P})$, we define its set of dominated linear previsions $\mathcal{M}(\overline{P})$ as

$$\mathcal{M}(\overline{P}) = \{P \in \mathcal{P}(\Omega) \mid (\forall X \in \mathcal{G})(P(X) \leq \overline{P}(X))\}.$$

We say that $(\Omega, \mathcal{G}, \overline{P})$ *avoids sure loss* iff $\mathcal{M}(\overline{P}) \neq \emptyset$, and is *coherent* iff it avoids sure loss and

$$\overline{P}(X) = \sup\{P(X) \mid P \in \mathcal{M}(\overline{P})\}, \quad X \in \mathcal{G}.$$

When $(\Omega, \mathcal{G}, \overline{P})$ avoids sure loss, its *natural extension* $(\Omega, \mathcal{L}(\Omega), \overline{E})$ to $\mathcal{L}(\Omega)$ is defined as

$$\overline{E}(X) = \sup\{P(X) \mid P \in \mathcal{M}(\overline{P})\}, \quad X \in \mathcal{L}(\Omega).$$

It is the greatest coherent upper prevision that is dominated by $\overline{P}$ on its domain $\mathcal{G}$. $(\Omega, \mathcal{G}, \overline{P})$ is coherent iff it coincides on its domain $\mathcal{G}$ with its natural extension. The natural extension $(\Omega, \mathcal{L}(\Omega), \underline{E})$ of the corresponding lower prevision $(\Omega, -\mathcal{G}, \underline{P})$ is defined by $\underline{E}(X) = -\overline{E}(-X) = \inf\{P(X) \mid P \in \mathcal{M}(\overline{P})\}$, $X \in \mathcal{L}(\Omega)$. Equivalent alternative definitions of avoiding sure loss, coherence and natural extension, with a direct behavioural interpretation, can be found in [19].

It has been shown elsewhere [8, 20] that a possibility measure Π on $(\Omega, \wp(\Omega))$ is a coherent upper probability iff Π is normal. Its dual necessity measure is then the corresponding coherent lower probability. It can be shown that their natural extensions to $\mathcal{L}(\Omega)$, respectively denoted by Π and N, are given by the following Riemann(-Stieltjes) integrals:

$$\Pi(X) = \int_{-\infty}^{+\infty} x \mathrm{d}\underline{F}_X(x) = \inf X + \int_{\inf X}^{\sup X} \sup\{\pi(\omega) \mid X(\omega) > x\} \mathrm{d}x \tag{1}$$

and

$$\mathrm{N}(X) = \int_{-\infty}^{+\infty} x \mathrm{d}\overline{F}_X(x) = \inf X + \int_{\inf X}^{\sup X} \inf\{\nu(\omega) \mid X(\omega) < x\} \mathrm{d}x. \tag{2}$$

In these expressions, $\underline{F}_X$ and $\overline{F}_X$ are respectively the *lower* and *upper distribution functions* of the gamble X w.r.t. the pair Π and N, defined by, for any $y \in \mathbb{R}$:

$$\underline{F}_X(y) = \mathrm{N}(\{\omega \in \Omega \mid X(\omega) \leq y\}) \text{ and } \overline{F}_X(y) = \Pi(\{\omega \in \Omega \mid X(\omega) \leq y\}).$$

In what follows, we show that any possibility measure and any necessity measure can be constructed using a probability measure and a nested multivalued mapping. In particular, we thereby retrieve and at the same time refine a result proven by Dubois and Prade [12] for the special case $\Omega = \mathbb{R}$. The line of reasoning we follow here draws from the ideas that lie at the basis of the Dempster-Shafer theory of evidence [10, 17] and elementary random set theory [15].

At the same time, we show that this construction allows us to extend possibility and necessity measures in a straightforward way to upper and lower previsions which will turn out to be precisely their natural extensions, discussed above. In this way, we arrive at alternative formulas for the calculation of the natural extension of possibility and necessity measures.

2. Multivalued mappings, gambles and events

Consider the real unit interval $[0,1]$ and a mapping Ψ from $[0,1]$ to $\wp(\Omega)$, also called a *multivalued mapping* from $[0,1]$ to Ω. We assume that Ψ is *antitone*:

$$(\forall(x,y)\in[0,1]^2)(x\geq y\Rightarrow \Psi(x)\subseteq\Psi(y)), \tag{C1}$$

which implies that the sets in $\Psi([0,1])$ are *nested*. We also want every element of Ω to be contained in some $\Psi(x)$, $x\in[0,1]$, which amounts to:

$$\Psi(0)=\Omega. \tag{C2}$$

For reasons that will become clear at the end of section 3, we do not exclude the existence of x in $[0,1]$ for which $\Psi(x)=\emptyset$. In order to deal with this, we define the set $\mathcal{E}_\Psi=\{x\in[0,1]\mid \Psi(x)=\emptyset\}$. By (C1), $\mathcal{E}_\Psi$ is an up-set [3] of the chain $([0,1],\leq)$. If we define $\varepsilon_\Psi=\inf\mathcal{E}_\Psi$, then $\mathcal{E}_\Psi=]\varepsilon_\Psi,1]$ or $\mathcal{E}_\Psi=[\varepsilon_\Psi,1]$.

With any gamble X on Ω, we associate two gambles X^* and X_* on $[0,1]$, defined as follows:

$$X^*(x)=\begin{cases}\sup_{\omega\in\Psi(x)}X(\omega) & ;\quad x\in[0,1]\setminus\mathcal{E}_\Psi\\ \theta^* & ;\quad x\in\mathcal{E}_\Psi\end{cases}$$

$$X_*(x)=\begin{cases}\inf_{\omega\in\Psi(x)}X(\omega) & ;\quad x\in[0,1]\setminus\mathcal{E}_\Psi\\ \theta_* & ;\quad x\in\mathcal{E}_\Psi,\end{cases}$$

where θ^* and θ_* are real numbers to be determined shortly. For $x\in\mathcal{E}_\Psi$, $\sup_{\omega\in\Psi(x)}=-\infty$ and $\inf_{\omega\in\Psi(x)}=+\infty$. Since we want X^* and X_* to be gambles on $[0,1]$, i.e., *bounded* $[0,1]-\mathbb{R}$-mappings, we try and remedy this by introducing θ^* and θ_*, which we determine by imposing extra conditions on X^* and X_*. First of all, we want that $(-X)^*=-X_*$, which is equivalent[1] to $\theta^*+\theta_*=0$. Also, for any event A in Ω, we find in particular that

$$A^*(x)=\begin{cases}\theta^* & ;\quad x\in\mathcal{E}_\Psi\\ 0 & ;\quad x\notin\mathcal{E}_\Psi \text{ and } \Psi(x)\cap A=\emptyset\\ 1 & ;\quad x\notin\mathcal{E}_\Psi \text{ and } \Psi(x)\cap A\neq\emptyset,\end{cases}$$

and similarly

$$A_*(x)=\begin{cases}\theta_* & ;\quad x\in\mathcal{E}_\Psi\\ 0 & ;\quad x\notin\mathcal{E}_\Psi \text{ and } \Psi(x)\not\subseteq A\\ 1 & ;\quad x\notin\mathcal{E}_\Psi \text{ and } \Psi(x)\subseteq A.\end{cases}$$

We want A^* and A_* to correspond to events in $[0,1]$. θ^* and θ_* may therefore only assume the values 0 and 1. $\theta^*+\theta_*=0$ then implies that $\theta^*=\theta_*=0$, whence, with obvious notations,

$$A^*=\{x\in[0,1]\mid\Psi(x)\cap A\neq\emptyset\} \text{ and } A_*=\{x\in[0,1]\mid\emptyset\neq\Psi(x)\subseteq A\}.$$

[1] This holds if $\mathcal{E}_\Psi\neq\emptyset$. If $\mathcal{E}_\Psi=\emptyset$, the introduction of θ^* and θ_* is not necessary.

Note that $A_* \subseteq A^*$, $A_* \cap \mathcal{E}_\Psi = \emptyset$ and $A^* \cap \mathcal{E}_\Psi = \emptyset$. Moreover, $(\mathrm{co}A)^* = \mathrm{co}\mathcal{E}_\Psi \setminus A_*$, and $(\mathrm{co}A)_* = \mathrm{co}\mathcal{E}_\Psi \setminus A^*$, where the complement 'co' on the left hand sides is relative to Ω, and on the right hand sides relative to $[0,1]$. In particular, $\emptyset_* = \emptyset^* = \emptyset$ and $\Omega_* = \Omega^* = \mathrm{co}\mathcal{E}_\Psi$.

The interpretation of these notions is straightforward. The multivalued mapping Ψ can be seen as a way to transmit information from $[0,1]$ to Ω [19]. For a gamble X on Ω, X_* is the smallest gamble on $[0,1]$ compatible with X, and X^* the greatest. That $X_*(x) = X^*(x) = 0$ if $\Psi(x) = \emptyset$ assures that the gambles X_* and X^* are (behaviourally) neutral in those elements of $[0,1]$ that do not connect to elements of Ω.

We now investigate the Borel measurability of the events A^* and A_*, and the gambles X^* and X_*. For any event A in Ω, A^* is a down-set [3] and $A_* \cup \mathcal{E}_\Psi$ an up-set of the chain $([0,1],\leq)$. As a consequence, if we define the following elements of $[0,1]$ for any $A \subseteq \Omega$:

$$\eta^*(A) = \sup A^* = \sup\{x \in [0,1] \mid \Psi(x) \cap A \neq \emptyset\} \leq \varepsilon_\Psi$$
$$\eta_*(A) = \inf A_* = \inf(A_* \cup \mathcal{E}_\Psi) = \inf\{x \in [0,1] \mid \emptyset \neq \Psi(x) \subseteq A\} \leq \varepsilon_\Psi,$$

we find that

$$\begin{aligned} A^* &= [0,\eta^*(A)[\text{ or } A^* = [0,\eta^*(A)] \\ A_* \cup \mathcal{E}_\Psi &= [\eta_*(A),1] \text{ or } A_* \cup \mathcal{E}_\Psi =]\eta_*(A),1]. \end{aligned} \tag{3}$$

To prove (3), remark that, if $x < \eta^*(A)$, the characterization of supremum on a chain tells us that there exists a y in A^* for which $x < y$ and therefore $x \in A^*$. A similar proof may be given for the second statement. Since we have seen above that $\mathcal{E}_\Psi =]\varepsilon_\Psi,1]$ or $\mathcal{E}_\Psi = [\varepsilon_\Psi,1]$, this implies that the sets A^* and A_* are subintervals of $[0,1]$ and therefore *Borel sets* on $[0,1]$. It should also be noted that $A_* = \emptyset$ implies $(\mathrm{co}A)^* = \mathrm{co}\mathcal{E}_\Psi$, and that $A_* \neq \emptyset$ implies $A^* = \mathrm{co}\mathcal{E}_\Psi$.

In order to investigate the Borel measurability of the gambles X^* and X_*, we must for instance check whether, for every y in $\mathbb{R}$, the sets $D_y^{X^*} = \{x \in [0,1] \mid X^*(x) \leq y\}$ and $S_y^{X_*} = \{x \in [0,1] \mid X_*(x) \geq y\}$ are Borel sets on $[0,1]$. Obviously,

$$D_y^{X^*} = \begin{cases} (D_y^X)_* & ; \quad y < 0 \\ (D_y^X)_* \cup \mathcal{E}_\Psi & ; \quad y \geq 0 \end{cases} \text{ and } S_y^{X_*} = \begin{cases} (S_y^X)_* & ; \quad y > 0 \\ (S_y^X)_* \cup \mathcal{E}_\Psi & ; \quad y \leq 0, \end{cases} \tag{4}$$

where $S_y^X = \{\omega \in \Omega \mid X(\omega) \geq y\}$ and $D_y^X = \{\omega \in \Omega \mid X(\omega) \leq y\}$. Since in these expressions $\mathcal{E}_\Psi$, $(D_y^X)_*$ and $(S_y^X)_*$ are subintervals of $[0,1]$, and therefore Borel measurable, we are led to the following proposition.

Proposition 1. *For any gamble X on Ω, X^* and X_* are Borel measurable gambles on* $[0,1]$. *For any subset A of Ω, A^* and A_* are Borel measurable subsets of* $[0,1]$.

3. Random sets, possibility and necessity measures

After these preliminary considerations, we are ready to proceed to the main topic of this paper. Consider a probability *measure* P_o on $([0,1],\mathcal{B}([0,1]))$, where $\mathcal{B}([0,1])$ is the σ-field of the Borel sets on $[0,1]$. In other words, P_o is a positive, *countably* additive set function defined on $\mathcal{B}([0,1])$ for which $P_o([0,1]) = 1$. Such a probability measure has a unique extension to a linear prevision $([0,1],\mathcal{K}(\mathcal{B}([0,1])),E_{P_o})$, where $\mathcal{K}(\mathcal{B}([0,1]))$ is the set of the $\mathcal{B}([0,1])$-measurable gambles on $[0,1]$, that is, the Borel measurable bounded $[0,1]-\mathbb{R}$-mappings [19]. As our notation suggests, this E_{P_o} is also the natural extension of the additive probability $([0,1],\mathcal{B}([0,1]),P_o)$ to $\mathcal{K}(\mathcal{B}([0,1]))$. Moreover, for any Borel measurable gamble Y on $[0,1]$:

$$E_{P_o}(Y) = \int_{[0,1]} Y \,\mathrm{d}P_o$$

and the integral in this expression is the Lebesgue integral w.r.t. the measure P_o [19]. Note that the bounded mapping Y is always integrable on the compactum $[0,1]$. We furthermore assume that the measure P_o is *absolutely continuous* w.r.t. the Lebesgue measure on $[0,1]$. Why this assumption is necessary, will become apparent in the proof of Lemma 3.

The introduction of the probability measure P_o on the measurable space $([0,1],\mathcal{B}([0,1]))$ allows us to formally interpret the multivalued mapping Ψ as a random variable, whose values are subsets of Ω. Ψ is therefore also called a *random set* [15], or a random subset of Ω.

We want to use the linear prevision E_{P_o} to construct a pair of coherent upper and lower previsions on $\mathcal{L}(\Omega)$, and consequently also a pair of coherent upper and lower probabilities on $\wp(\Omega)$. We assume that

$$P_o(\mathrm{co}\mathcal{E}_\Psi) > 0, \text{ or equivalently, } P_o(\mathcal{E}_\Psi) < 1. \tag{C3}$$

In other words, we exclude that the random set Ψ is empty a.s. (P_o). Proposition 1 then enables us to define an *upper prevision* $(\Omega,\mathcal{L}(\Omega),\Pi_\Psi)$ as follows. For any X in $\mathcal{L}(\Omega)$:

$$\Pi_\Psi(X) = E_{P_o}(X^*)/P_o(\mathrm{co}\mathcal{E}_\Psi) = \int_{[0,1]} X^*\mathrm{d}P_o/P_o(\mathrm{co}\mathcal{E}_\Psi).$$

Since we have made sure that $X_* = -(-X)^*$, we find for the corresponding *lower prevision* $(\Omega,\mathcal{L}(\Omega),\mathrm{N}_\Psi)$, with $\mathrm{N}_\Psi(X) = -\Pi_\Psi(-X)$, that

$$\mathrm{N}_\Psi(X) = E_{P_o}(X_*)/P_o(\mathrm{co}\mathcal{E}_\Psi) = \int_{[0,1]} X_*\mathrm{d}P_o/P_o(\mathrm{co}\mathcal{E}_\Psi).$$

In what follows, we intend to study these upper and lower previsions in more detail.

For a start, we find the following expressions for the corresponding *upper and lower probability* of the event A in Ω:

$$\begin{aligned}
\Pi_\Psi(A) &= P_o(A^*)/P_o(\mathrm{co}\mathcal{E}_\Psi) = P_o(\{x\in[0,1] \mid \Psi(x)\cap A \neq \emptyset\})/P_o(\mathrm{co}\mathcal{E}_\Psi)\\
\mathrm{N}_\Psi(A) &= P_o(A_*)/P_o(\mathrm{co}\mathcal{E}_\Psi) = P_o(\{x\in[0,1] \mid \emptyset \neq \Psi(x)\subseteq A\})/P_o(\mathrm{co}\mathcal{E}_\Psi)\\
\mathrm{N}_\Psi(A) &= 1 - \Pi_\Psi(\mathrm{co}A).
\end{aligned} \tag{5}$$

These are instances of the more general formulas proposed by Dempster in his paper on upper and lower probabilities induced by a multivalued mapping [10].

In particular, $\Pi_\Psi(\emptyset) = \mathrm{N}_\Psi(\emptyset) = 0$ and $\Pi_\Psi(\Omega) = \mathrm{N}_\Psi(\Omega) = 1$. Let us now show that the upper probability $(\Omega,\wp(\Omega),\Pi_\Psi)$ is a normal possibility measure, and that the lower probability $(\Omega,\wp(\Omega),\mathrm{N}_\Psi)$ is consequently a normal necessity measure. Consider any family $(A_j \mid j\in J)$ of subsets of Ω, then for any x in $[0,1]$:

$$\Psi(x)\cap\big(\bigcup_{j\in J} A_j\big) \neq \emptyset \Leftrightarrow (\exists j\in J)(\Psi(x)\cap A_j \neq \emptyset),$$

whence $(\bigcup_{j\in J} A_j)^* = \bigcup_{j\in J} A_j^*$ and therefore also $\eta^*(\bigcup_{j\in J} A_j) = \sup_{j\in J}\eta^*(A_j)$. If we combine this with (3), we find that

$$\big(\bigcup_{j\in J} A_j\big)^* = [0,\sup_{j\in J}\eta^*(A_j)[\text{ or } \big(\bigcup_{j\in J} A_j\big)^* = [0,\sup_{j\in J}\eta^*(A_j)].$$

Using Lemma 3, we find the following important result.

Theorem 2. *The restriction of* Π_Ψ *to* $\wp(\Omega)$ *is a normal possibility measure and the restriction of* N_Ψ *to* $\wp(\Omega)$ *a normal necessity measure on* $(\Omega,\wp(\Omega))$*. For their respective distributions* π_Ψ *and* ν_Ψ *we find that, for any* ω *in* Ω*:*

$$\begin{aligned}
\pi_\Psi(\omega) &= P_o(\{x\in[0,1] \mid \omega\in\Psi(x)\})/P_o(\mathrm{co}\mathcal{E}_\Psi)\\
\nu_\Psi(\omega) &= P_o(\{x\in[0,1] \mid \omega\notin\Psi(x)\neq\emptyset\})/P_o(\mathrm{co}\mathcal{E}_\Psi).
\end{aligned} \tag{6}$$

The upper probability $(\Omega,\wp(\Omega),\Pi_\Psi)$ *and the lower probability* $(\Omega,\wp(\Omega),\mathrm{N}_\Psi)$ *are therefore coherent.*

Lemma 3. *Consider a family* $(x_j \mid j \in J)$ *of elements of* $[0,1]$ *and an element* x *of* $[0,1]$.

1. $P_o([0,x[) = P_o([0,x])$.
2. $P_o([0,\sup_{j\in J} x_j]) = \sup_{j\in J} P_o([0,x_j])$.

Proof. The proof of the first statement is trivial, since P_o is by assumption absolutely continuous w.r.t. the Lebesgue measure on $[0,1]$. Let us therefore prove the second statement. Since P_o is increasing, it need only be shown that $\sup_{j\in J} P_o([0,x_j]) \geq P_o([0,\sup_{j\in J} x_j])$. If $\sup_{j\in J} x_j$ belongs to the family $(x_j \mid j \in J)$, the proof is immediate. Let us therefore assume that $(\forall i \in J)(x_i < \sup_{j\in J} x_j)$, which implies that there exists a strictly increasing sequence $(x_n^*)_{n\in\mathbb{N}}$ in the family $(x_j \mid j \in J)$ which converges to $\sup_{j\in J} x_j$. As a consequence, the strictly increasing sequence $([0,x_n^*])_{n\in\mathbb{N}}$ converges to $\bigcup_{n\in\mathbb{N}}[0,x_n^*] = [0,\sup_{j\in J} x_j[$, and therefore, taking into account the well-known limit properties of a measure:

$$\sup_{j\in J} P_o([0,x_j]) \geq \sup_{n\in\mathbb{N}} P_o([0,x_n^*]) = \lim_{n\to+\infty} P_o([0,x_n^*]) = P_o(\bigcup_{n\in\mathbb{N}}[0,x_n^*]) = P_o([0,\sup_{j\in J} x_j[).$$

This proves the second statement, also taking into account the first. □

In what follows, we shall call the couple (P_o,Ψ) a *random set representation* of the possibility measure Π_Ψ and the necessity measure N_Ψ on $(\Omega,\wp(\Omega))$.

We now proceed to show that *every normal possibility and necessity measure* on $(\Omega,\wp(\Omega))$ can be obtained in this way, i.e., have a random set representation! Indeed, given any normal possibility measure Π on $(\Omega,\wp(\Omega))$, define the multivalued mapping Ψ as follows: for any x in $[0,1]$, $\Psi(x) = S_x^\pi = \{\omega\in\Omega \mid \pi(\omega)\geq x\}$, where π is the distribution of Π. This Ψ clearly satisfies conditions (C1) and (C2). Furthermore, for any ω in Ω, $\{\omega\}^* = [0,\pi(\omega)]$. If we therefore let P_o be the Lebesgue measure λ on $[0,1]$, its is clear that $P_o(\{\omega\}^*) = \pi(\omega)$. Finally, since Π is normal, $\sup_{\omega\in\Omega}\pi(\omega) = 1$, whence $\mathcal{E}_\Psi = \emptyset$ or $\mathcal{E}_\Psi = \{1\}$, according to whether the supremum 1 of π is reached or not. In any case, $P_o(\mathrm{co}\mathcal{E}_\Psi) = 1$ and (C3) is satisfied. This proves our assertion. The couple $(\lambda, S_\cdot^\pi)$ will be called the *standard* random set representation of the possibility measure Π and its dual necessity measure N.

We want to stress that that for this choice of Ψ, $\mathcal{E}_\Psi$ is not necessarily empty! In the case $\mathcal{E}_\Psi = \{1\}$, π does not reach its supremum in any of the points of its domain. The distribution of Π is then called *nonmodal*. It therefore turns out that we had to allow $\mathcal{E}_\Psi \neq \emptyset$ in order to be able to incorporate into the random set model *normal* possibility measures with *nonmodal* distributions.

4. Random sets and natural extension

Let us now use this information to derive a formula for the calculation of $\Pi_\Psi(X)$ and $\mathrm{N}_\Psi(X)$, $X\in\mathcal{L}(\Omega)$. We know that by definition, and also using a well-known result from probability theory, since X^* is Borel measurable:

$$\Pi_\Psi(X) = \int_{[0,1]} X^* \mathrm{d}P_o / P_o(\mathrm{co}\mathcal{E}_\Psi) = \int_{\mathbb{R}} \mathbf{1}_{\mathbb{R}} \mathrm{d}F_{X^*} / P_o(\mathrm{co}\mathcal{E}_\Psi).$$

In this expression, $\mathbf{1}_{\mathbb{R}}$ is the identical permutation of the reals, and the integral on the right hand side is the Lebesgue-Stieltjes integral associated with the probability distribution function F_{X^*} of the gamble X^*, i.e., for any y in $\mathbb{R}$: $F_{X^*}(y) = P_o(\{x\in[0,1] \mid X^*(x)\leq y\}) = P_o(D_y^{X^*})$. Now, taking into account (4) and (C3), we find for any y in $\mathbb{R}$,

$$F_{X^*}(y) = P_o(\mathrm{co}\mathcal{E}_\Psi)\cdot\begin{cases}\underline{F}_X(y) & ; \quad y<0\\ \underline{F}_X(y) + \dfrac{P_o(\mathcal{E}_\Psi)}{P_o(\mathrm{co}\mathcal{E}_\Psi)} & ; \quad y\geq 0,\end{cases}$$

where $\underline{F}_X$ is the lower distribution function of X associated with the pair of upper and lower probabilities Π_Ψ and N_Ψ. Note that $\underline{F}_X$ is nonconstant at most on a bounded real interval, because the gamble X is by definition bounded. This essentially reduces the integration domain $\mathbb{R}$ to a bounded interval. Since moreover $\mathbf{1}_{\mathbb{R}}$ is continuous, the above-mentioned Lebesgue-Stieltjes integral is equal to the corresponding Riemann-Stieltjes integral [14], whence, with obvious notations, and taking into account well-known results from the theory of Riemann-Stieltjes integration [1]:

$$\Pi_\Psi(X) = \int_{-\infty}^{+\infty} x \mathrm{d}\underline{F}_X(x) + 0 \cdot \frac{P_o(\mathcal{E}_\Psi)}{P_o(\mathrm{co}\mathcal{E}_\Psi)} = \int_{-\infty}^{+\infty} x \mathrm{d}\underline{F}_X(x),$$

and consequently also

$$\mathrm{N}_\Psi(X) = \int_{-\infty}^{+\infty} x \mathrm{d}\overline{F}_X(x),$$

where $\overline{F}_X$ is the upper distribution function of X associated with the pair Π_Ψ and N_Ψ. In other words, using the probability measure P_o and the nested multivalued mapping Ψ, we are able to construct in a natural way not only possibility and necessity measures, but also their natural extensions! As a corollary, we find that the upper and lower previsions $(\Omega, \mathcal{L}(\Omega), \Pi_\Psi)$ and $(\Omega, \mathcal{L}(\Omega), \mathrm{N}_\Psi)$ are coherent.

This result, together with the argument presented at the end of the previous section, also allows us to give an alternative expression for the natural extensions of a possibility measure Π and its dual necessity measure N on $(\Omega, \wp(\Omega))$. Let π be the distribution of Π. For the standard random set representation $(P_o, \Psi) = (\lambda, S^\pi_\cdot)$ of Π and N, we know that $P_o(\mathrm{co}\mathcal{E}_\Psi) = 1$, whence, if we also denote by Π the natural extension of Π, $\Pi(X) = \int_{[0,1]} X^* \mathrm{d}\lambda$, where in particular for $x \in [0, 1[$, $X^*(x) = \sup\{X(\omega) \mid \pi(\omega) \geq x\}$. Since X^* is decreasing on $[0, 1[$, it is discontinuous in at most a countable number of elements of $[0, 1]$, and therefore the Lebesgue integral is equal to the corresponding Riemann integral [2]:

$$\Pi(X) = \int_0^1 \sup\{X(\omega) \mid \pi(\omega) \geq x\} \mathrm{d}x = \int_0^1 \sup\{X(\omega) \mid \pi(\omega) > x\} \mathrm{d}x.$$

The last equality holds because the two integrands may only differ in their points of discontinuity. Similarly, if we also denote by N the natural extension of N:

$$\begin{aligned}\mathrm{N}(X) &= \int_0^1 \inf\{X(\omega) \mid \pi(\omega) \geq x\} \mathrm{d}x = \int_0^1 \inf\{X(\omega) \mid \pi(\omega) > x\} \mathrm{d}x \\ &= \int_0^1 \inf\{X(\omega) \mid \nu(\omega) \leq x\} \mathrm{d}x = \int_0^1 \inf\{X(\omega) \mid \nu(\omega) < x\} \mathrm{d}x.\end{aligned}$$

When we compare these expressions with (1) and (2), the symmetry (exchangeability) between π and ν on the one hand, and X on the other hand, is more than striking. Note that an analogous symmetry exists in the probabilistic case between X and the probability distribution (density).

5. Conclusion

We have shown that any possibility and necessity measure can be constructed using a probability measure and a multivalued mapping (a random set). At the same time, we have proven that their natural extensions can be obtained in a similar way. This course of reasoning has provided us with alternative formulas for the calculation of these natural extensions.

Acknowledgements

Gert de Cooman is a Postdoctoral Fellow of the Belgian National Fund for Scientific Research (NFWO). He would like to thank the NFWO for partially funding the research reported on in this paper.

This paper presents research results of the Belgian Programme on Interuniversity Poles of Attraction initiated by the Belgian state, Prime Minister's Office for Science, Technology and Culture. The scientific responsibility rests with the authors.

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