Robust Inventory Management Using Tractable Replenishment Policies

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Abstract

We propose tractable replenishment policies for a multi-period, single product inventory control problem under ambiguous demands, that is, only limited information of the demand distributions such as mean, support and deviation measures are available. We obtain the parameters of the tractable replenishment policies by solving a deterministic optimization problem in the form of second order cone optimization problem (SOCP). Our framework extends to correlated demands and is developed around a factor-based model, which has the ability to incorporate business factors as well as time series forecast effects of trend, seasonality and cyclic variations. Computational results show that with correlated demands, our model outperforms a state independent base-stock policy derived from dynamic programming and an adaptive myopic policy.

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1 Introduction

Inventory ties up working capital and incurs holding costs, reducing profit every day excess stock is held. Good inventory management has hence become crucial to businesses as they seek to continually improve their customer service and profit margins, in the heat of global competition and demand variability. Baldenius and Reichelstein [3] offered perhaps the most convincing study of the contribution of good inventory management to profitability. They studied inventories of publicly traded American manufacturing companies between 1981 and 2000, and they concluded that "Firms with abnormally high inventories have abnormally poor long-term stock returns. Firms with slightly lower than average inventories have good stock returns, but firms with the lowest inventories have only ordinary returns".

The ability to incorporate more realistic assumptions about product demand into inventory models is one key factor to profitability. Practical models of inventory would need to address the issue of demand forecasting while staying sufficiently immunized against uncertainty and maintaining tractability. In most industrial contexts, demand is uncertain and hard to forecast. Many demand histories have factors that behave like random walks that evolve over time with frequent changes in their directions and rates of growth or decline. In practice, for such demand processes, inventory managers often rely on forecasts based on a time series of prior demand, which are often correlated over time. For example, demand may depend on factors such as market outlook, oil prices and so forth, and contains effects of trend, seasonality, cyclic variation and randomness.

In this paper, we address the problem of optimizing multi-period inventory using factor-based demand models, where the coefficients of the factors can be forecasted statistically, perhaps using historical time-series data. We assume that the demands may be correlated and are ambiguous, that is, limited information of the demand distributions (only the mean, support and deviation measures) are available. Using robust optimization techniques, we developed a tractable methodology that uses past demand realizations to adaptively control multi-period inventory. Our model also includes a range of features such as delivery delay and capacity limit on order quantity.

Our work is closely related to the multi-period newsvendor problem, a well studied problem in operations research. For the single product newsvendor, it is well-known that the base-stock policy based on a critical fractile is optimal. See Scarf [36, 37] and Zipkin [40]. Azoury [2] and Miller [32] later extended the results to other classes of distribution. For correlated demands, Veinott [39], characterized conditions under which a myopic policy is optimal. Extending the results of Veinott, Johnson and Thompson [28] considered an autoregressive, moving-average (ARMA) demand process, zero replenishment lead-time and no backlogs, and showed the optimality of a myopic policy when there demand in each period is bounded. Lovejoy [30] showed that a myopic critical-fractile policy is optimal or near optimal in some inventory models with parameter adaptive demand processes, citing exponential smoothing on the demand process and bayesian updating on uniformly distributed demand as examples. Song and Zipkin [38] addressed the case of Poisson demand, where the transition rate between states is governed by a Markov process.

Sampling-based approximation have also been applied to the newsvendor problem. The key idea is to represent uncertainty by a finite sample of scenarios and solve the sample average approximation counterpart of the problem. While there is no requirement to know the full demand distribution, it is assumed that N independent samples of the demand drawn from the true distribution are available. The key to the quality of the solution is the number of samples. Levi, Roundy, and Shmoys [31] gave theoretical results on the sample size required to achieve ϵ-optimal solution. They showed that when the sample size is greater than $\frac{9}{2\epsilon^2}\left(\frac{\min(b,h)}{b+h}\right)^{-2}\ln(\frac{2}{\delta})$, the solution of the sample average approximation is at most $1+\epsilon$ times the optimal solution with probability of at least $1-\delta$, with b being the backlog cost and h being the holding cost. Other sampling-based approaches include infinitesimal perturbation analysis (see Glasserman et. al. [25]) which uses stochastic gradient estimation technique, and the concave adaptive value estimation procedure, which successively approximates the objective cost function with a sequence of piecewise linear functions (see Godfrey et. al., [26] and Powell et. al. [34]).

When only partial information on the underlying demand distribution is known, the problem is commonly known as the distribution-free newsvendor model. Research on distribution-free inventory control dates back to Scarf [35], where he considered a single period newsvendor problem and determined the orders that minimize the maximum expected cost over all possible demand distributions with the same first and second moments. Subsequent work on distribution-free newsvendor problem includes Gallego and Moon [23], and Gallego, Ryan and Simchi-Levi [24], where they showed that a base-stock policy is optimal for the multi-period newsvendor problem with discrete demand distributions.

More recently, Bertsimas and Thiele [13], Ben-Tal et.al. [4], and Beinstock and Ozbay [15] developed new approaches to the distribution-free inventory control problem, which has the advantage of being more tractable than sampling-based approaches. They employed robust optimization techniques to guarantee the feasibility of the obtained solution for all possible values of the uncertain parameters in the designated uncertainty set. The main idea in robust optimization is to immunize uncertain mathematical optimization against infeasibility while preserving the tractability of the model, see Ben-Tal and Nemirovski [5, 6, 7], Bertsimas and Sim [11, 12], Bertsimas, et al. [10], El-Ghaoui and Lebret

[20], and El-Ghaoui, et al. [21]. Our proposed model is related to this track of research, which has been gaining substantial popularity as a useful methodology for handling uncertainty.

The paper is organized as follows. In Section 2 we describe a stochastic inventory model. We formulate our robust inventory models in Section 3 and discuss extensions in Section 4. We describe computational results in Section 5 and conclude the paper in Section 6.

Notations: Throughout this paper, we denote a random variable with the tilde sign such as $\tilde{y}$ and vectors with bold face lower case letters such as $\boldsymbol{y}$. We use $\mathbf{y}'$ to denote the transpose of vector $\mathbf{y}$. Also, denote $y^+ = \max(y, 0)$, $y^- = \max(-y, 0)$, and $\|\boldsymbol{y}\|_2 = \sqrt{\sum y_i^2}$.

2 Stochastic Inventory Model

The stochastic inventory model involves the derivation of replenishment decisions over a discrete planning horizon consisting of a finite number of periods under stochastic demand. The demand for each period is usually a sequence of random variables which are *not* necessarily identically distributed and *not* necessarily independent. We consider an inventory system with T planning horizons from $t = 1$ to $t = T$. External demands arrive at the inventory system and the system replenishes its inventory from some central warehouse (or supplier) with ample supply. The time line of events is as follows:

1. At the beginning of the tth time period, before observing the demand, the inventory manager places an order of x_t at unit cost c_t for the product to be arrived after a (fixed) order lead-time of L periods. Orders placed at the *beginning* of the tth time period will arrive at the *beginning* of $t + L$th period. We assume that replenishment ceases at the end of the planning horizon, so that the last order is placed in period $T - L$. Without loss of generality, we assume that purchase costs for inventory are charged at the time of order. The case where purchase costs are charged at the time of delivery can be represented by a straight-forward shift of cost indices.

2. At the beginning of each time period t, the inventory manager faces an initial inventory position y_t and receives an order of x_{t-L}. The demand of inventory for the period is realized at the end of the time period. After receiving a demand of d_t, the inventory position at the end of the period is $y_t + x_{t-L} - d_t$.

3. Excess inventory is carried to the next period incurring a per-unit overage (holding) cost. On the other hand, each unit of unsatisfied demand is backlogged (carried over) to the next period with

a per-unit underage (backlogging) penalty cost. At the last period, $t = T$, the cost penalty for unmet demand can be accounted through the underage penalty cost.

We assume a risk neutral inventory manager whose objective is to determine the dynamic ordering quantities x_t from period $t = 1$ to period $t = T - L$ so as to minimize the total expected ordering, inventory overage (holding) and underage (backlog) costs in response to the uncertain demands. Observe that for $L \geq 1$, the quantities x_{t-L}, $t = 1, \ldots, L$ are known values. They denote orders made before period $t = 1$ and are inventories in the delivery pipeline when the planning horizon starts.

We introduce the following notations.

- $\tilde{d}_t$: stochastic exogenous demand at period t
- $\tilde{\boldsymbol{d}}_t$: a vector of random demands from period 1 to t, that is, $\tilde{\boldsymbol{d}}_t = (\tilde{d}_1, \ldots, \tilde{d}_t)$
- $x_t(\tilde{\boldsymbol{d}}_{t-1})$: order placed at the beginning of the tth time period after observing $\tilde{\boldsymbol{d}}_{t-1}$. The first period inventory order is denoted by $x_1(\tilde{\boldsymbol{d}}_0) = x_1^0$
- $y_t(\tilde{\boldsymbol{d}}_{t-1})$: inventory position at the beginning of the tth time period
- h_t: unit inventory overage (holding) cost charged on excess inventory at the end of the tth time period
- b_t: unit underage (backlog) cost charged on backlogged inventory at the end of the tth time period
- c_t: unit purchase cost of inventory for orders placed at the tth time period
- S_t: the maximum amount that can be ordered at the tth time period.

We use $x_t(\tilde{\boldsymbol{d}}_{t-1})$ to represent the non-anticipative replenishment policy at the beginning of period t. That is, the replenishment decision is based solely on the observed information available at the beginning of period t, which is the demand vector $\tilde{\boldsymbol{d}}_{t-1} = (\tilde{d}_1, \ldots, \tilde{d}_{t-1})$. Given the order quantity $x_{t-L}(\tilde{\boldsymbol{d}}_{t-L-1})$ and stochastic exogenous demand $\tilde{d}_t$, the inventory position at the *end* of the t time period (which is also the inventory position at start of $t + 1$ time period) is given by

$$y_{t+1}(\tilde{\boldsymbol{d}}_t) = y_t(\tilde{\boldsymbol{d}}_{t-1}) + x_{t-L}(\tilde{\boldsymbol{d}}_{t-L-1}) - \tilde{d}_t, \qquad t = 1, \ldots, T. \tag{1}$$

In resolving the initial boundary conditions, we adopt the following notations:

- The initial inventory position of the system is $y_1(\tilde{\boldsymbol{d}}_0) = y_1^0$.

- When $L \geq 1$, the orders that are placed before the planning horizon starts are denoted by

$$x_t(\tilde{\boldsymbol{d}}_{t-1}) = x_t^0, \qquad t = 1-L, \ldots, 0.$$

Note that Equation (1) can be written using the cumulative demand up to period t and cumulative order received as follows,

$$y_{t+1}(\tilde{\boldsymbol{d}}_t) = \underbrace{y_1^0}_{\text{initial inventory}} + \underbrace{\sum_{\tau=1}^{\min\{L,t\}} x_{\tau-L}^0}_{\text{committed orders}} + \underbrace{\sum_{\tau=L+1}^{t} x_{\tau-L}(\tilde{\boldsymbol{d}}_{\tau-L-1})}_{\text{order decisions}} - \underbrace{\sum_{\tau=1}^{t} \tilde{d}_\tau}_{\text{cumulative demands}}. \tag{2}$$

Observe that positive (respectively negative) value of $y_{t+1}(\tilde{\boldsymbol{d}}_t)$ represents the total amount of inventory overage (respectively underage) at the end of the period t after meeting demand. Thus, the total expected cost, including ordering, inventory overage and underage charges is equal to

$$\sum_{t=1}^{T}\left(\mathrm{E}\left(c_t x_t(\tilde{\boldsymbol{d}}_{t-1})\right) + \mathrm{E}\left(h_t(y_{t+1}(\tilde{\boldsymbol{d}}_t))^+\right) + \mathrm{E}\left(b_t(y_{t+1}(\tilde{\boldsymbol{d}}_t))^-\right)\right).$$

Therefore, the multi-period inventory problem can be formulated as a T stage stochastic optimization model as follows,

$$\begin{aligned} Z_{STOC} = \min \quad & \sum_{t=1}^{T}\left(\mathrm{E}\left(c_t x_t(\tilde{\boldsymbol{d}}_{t-1})\right) + \mathrm{E}\left(h_t(y_{t+1}(\tilde{\boldsymbol{d}}_t))^+\right) + \mathrm{E}\left(b_t(y_{t+1}(\tilde{\boldsymbol{d}}_t))^-\right)\right). \\ \text{s.t.} \quad & y_{t+1}(\tilde{\boldsymbol{d}}_t) = y_t(\tilde{\boldsymbol{d}}_{t-1}) + x_{t-L}(\tilde{\boldsymbol{d}}_{t-L-1}) - \tilde{d}_t \qquad t = 1, \ldots, T \\ & 0 \leq x_t(\tilde{\boldsymbol{d}}_{t-1}) \leq S_t \qquad t = 1, \ldots, T-L \end{aligned} \tag{3}$$

The aim of the stochastic optimization model is to derive a feasible replenishment policy that minimizes the expected ordering and inventory costs. That is, we seek a sequence of action rules that advises the inventory manager the action to take in time t as a function of the current inventory level and historical demand realizations. Unfortunately, the decision variables in Problem (3), $x_t(\tilde{\boldsymbol{d}}_{t-1}), t = 1 \ldots T-L$ and $y_t(\tilde{\boldsymbol{d}}_{t-1})$, $t = 2 \ldots T+1$ are functionals, which means Problem (3) is an optimization problem with infinite number of variables and constraints, and hence generally intractable.

The stochastic optimization problem (3) can also be formulated as a dynamic programming problem. For simplicity, assuming zero lead-time and no ordering capacity limit, the dynamic programming requires the following updates on the value function:

$$\begin{aligned} J_t(y_t, d_1, \ldots, d_{t-1}) \quad = \quad & \min_{x \geq 0} \mathrm{E}\Big(c_t x + r_t(y_t + x - \tilde{d}_t) + \\ & J_{t+1}(y_t + x - \tilde{d}_t, d_1, \ldots, d_{t-1}, \tilde{d}_t) \mid \tilde{d}_1 = d_1, \ldots, \tilde{d}_{t-1} = d_{t-1}\Big), \end{aligned}$$

where $r_t(u) = h_t \max(u, 0) + b_t \max(-u, 0)$. Maintaining the value function $J_t(\cdot)$ is computationally prohibitive, and hence most inventory control literatures identify conditions such that the value functions are not dependent on past demand history, so that the state space is computationally amenable. For instance, it is well known that when the lead-time is zero and the demands are independently distributed across time periods, there exists base-stock levels, q_t such that the following replenishment policy

$$x_t(\tilde{\boldsymbol{d}}_{t-1}) = (q_t - y_t)^+$$

is optimal. Note that in order to obtain an optimal state independent base-stock policy for positive lead-time, $L > 0$, we require some restrictions on the cost parameters. See for instance, Zipkin [40].

3 Robust Inventory Model

One of the central problems in stochastic models with unknown distribution is how to properly account for data uncertainty while maintaining tractability. Following the recent development of robust optimization such as Chen, Sim and Sun [17], Chen, Sim, Sun and Zhang [18] and Chen and Sim [19], we relax the assumption of full distributional knowledge and modify the representation of data uncertainties with the aim of producing a computationally tractable model. We adopt the factor-based demand model in which the demands are affinely dependent on some random primitive factors.

Factor-based Demand Model

We introduce a factor-based demand model in which the uncertain demand are affinely dependent on zero mean random factors $\tilde{\boldsymbol{z}} \in \Re^N$ as follows:

$$d_t(\tilde{\boldsymbol{z}}) \triangleq \tilde{d}_t = d_t^0 + \sum_{k=1}^{N} d_t^k \tilde{z}_k, \qquad t = 1, \ldots, T,$$

where

$$d_t^k = 0 \qquad \forall k \geq N_{t+1},$$

and $1 \leq N_1 \leq N_2 \leq \ldots \leq N_T = N$. Under a factor-based demand model, the random factors, $\tilde{z}_k$, $k = 1, \ldots, N$ are realized sequentially. At period t, the factors, $\tilde{z}_k$, $k = 1, \ldots, N_t$ has already been unfolded. In progressing to period $t+1$, the new factors $\tilde{z}_k$, $k = N_t + 1, \ldots, N_{t+1}$ are made available.

Demand that is affected by random noise or shocks can be represented by the factor-based demand model. For independently distributed demand, which is assumed in most inventory models, we have

$$d_t(\tilde{\boldsymbol{z}}) = d_t^0 + \tilde{z}_t, \qquad t = 1, \ldots, T,$$

in which $\tilde{z}_t$ are independently distributed. However, in many industrial contexts, demands across periods may be correlated. In fact, many demand histories behave more like random walks over time with frequent changes in directions and rate of growth or decline. See Johnson and Thompson [28] and Graves [27]. In those settings, we may consider standard forecasting techniques such as an ARMA(p, q) demand process (see Box et al. [16]) as follows

$$d_t(\tilde{\boldsymbol{z}}) = \begin{cases} d_t^0 & \text{if } t \leq 0 \\ \displaystyle\sum_{j=1}^{p} \phi_i d_{t-i}(\tilde{\boldsymbol{z}}) + \tilde{z}_t + \sum_{j=1}^{\min\{q,t-1\}} \theta_i \tilde{z}_{t-j} & \text{otherwise} \end{cases}$$

for some constants coefficients $\phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q$. Indeed, its is easy to show by induction that $d_t(\tilde{\boldsymbol{z}})$ can be expressed in the form of a factor-based demand model. Song and Zipkin [38] presented a world driven demand model where the demand is Poisson with rate controlled by finite Markov states representing different business environment. Unfortunately, it is difficult to determine exhaustively the business states and its state transition probabilities. On the other hand, factor-based models have been used extensively in finance for modeling returns as affine functions of external factors, in which the coefficients of the factors can be determined statistically. In the same way, we can apply the factor-based demand model to characterize the influence of demands with external factors such as market outlook, oil prices and so forth. Effects of trend, seasonality, cyclic variation, and randomness can also be incorporated.

Instead of assuming full distributions on the factors, which is practically prohibitive, we adopt a modest distributional assumption on the random factors, such as known means, supports and some aspects of deviations. The factors may be partially characterized using the forward and backward deviations, which are recently introduced by Chen, Sim and Sun [17].

Definition 1 *Given a random variable $\tilde{z}$ with zero mean, the forward deviation is defined as*

$$\sigma_f(\tilde{z}) \triangleq \sup_{\theta > 0} \left\{ \sqrt{2 \ln(\mathrm{E}(\exp(\theta \tilde{z})))/\theta^2} \right\} \tag{4}$$

and backward deviation is defined as

$$\sigma_b(\tilde{z}) \triangleq \sup_{\theta > 0} \left\{ \sqrt{2 \ln(\mathrm{E}(\exp(-\theta \tilde{z})))/\theta^2} \right\}. \tag{5}$$

Given a sequence of independent samples, we can essentially estimate the magnitude of the deviation measures from (4) and (5). Some of the properties of the deviation measures include:

Proposition 1 *(Chen, Sim and Sun [17])*
Let σ, p and q be respectively the standard, forward and backward deviations of a random variable, $\tilde{z}$ with zero mean.

(a) *Then $p \geq \sigma$ and $q \geq \sigma$. If $\tilde{z}$ is normally distributed, then $p = q = \sigma$.*

(b) *For all $\theta \geq 0$,*

$$\mathrm{P}(\tilde{z} \geq \theta p) \leq \exp(-\theta^2/2);$$

$$\mathrm{P}(\tilde{z} \leq -\theta q) \leq \exp(-\theta^2/2).$$

Proposition 1(a) shows that the forward and backward deviations are no less than the standard deviation of the underlying distribution, and under normal distribution, these two values coincide with the standard deviation. As exemplified in Proposition 1(b), the deviation measures provide an easy bound on the distributional tails.

The forward and backward deviations can be bounded from the support of $\tilde{z}$ as follows:

Theorem 1 *(Chen, Sim and Sun [17]) If $\tilde{z}$ has zero mean and distributed in $[-\underline{z}, \bar{z}]$, $\underline{z}, \bar{z} > 0$, then*

$$\sigma_f(\tilde{z}) \leq \bar{\sigma}_f(\tilde{z}) = \frac{\underline{z} + \bar{z}}{2} \sqrt{g\left(\frac{\underline{z} - \bar{z}}{\underline{z} + \bar{z}}\right)}$$

and

$$\sigma_b(\tilde{z}) \leq \bar{\sigma}_b(\tilde{z}) = \frac{\underline{z} + \bar{z}}{2} \sqrt{g\left(\frac{\bar{z} - \underline{z}}{\underline{z} + \bar{z}}\right)},$$

where

$$g(\mu) = 2 \max_{s>0} \left\{ \frac{\phi_\mu(s) - \mu s}{s^2} \right\},$$

and

$$\phi_\mu(s) = \ln\left(\frac{e^s + e^{-s}}{2} + \frac{e^s - e^{-s}}{2}\mu \right).$$

Moreover, the bounds are tight.

Note that the forward and backward deviations may be infinite for heavier-tailed distributions. The advantage of using the forward and backward deviations is the ability to capture distributional asymmetry and stochastic independence, while keeping the resultant optimization model computationally amicable. The interested reader may refer to Natarajan et al. [33] for the computational experience of using the forward and backward deviations in minimizing the Value-at-Risk of a portfolio, which gives surprisingly good out-of-sample performance on real-life data.

In this paper, we adopt the random factor model introduced by Chen and Sim [19], which encompasses most of the uncertainty models found in the literatures of robust optimization.

Assumption U: We assume that the random factors $\{\tilde{z}_j\}_{j=1:N}$ are zero mean random variables, with positive definite covariance matrix, $\mathbf{\Sigma}$. Let $\mathcal{W} = \{\boldsymbol{z} : -\underline{\boldsymbol{z}} \leq \boldsymbol{z} \leq \bar{\boldsymbol{z}}\}$ denotes the smallest compact convex set containing the support of $\tilde{\boldsymbol{z}}$. We denote a subset, $\mathcal{I} \subseteq \{1, \ldots, N\}$, which can be an empty set, such that $\tilde{z}_j,\ j \in \mathcal{I}$ are stochastically independent. Moreover, the corresponding forward and backward deviations (or their bounds used in Theorem 1) are given by $p_j = \sigma_f(\tilde{z}_j) > 0$ and $q_j = \sigma_b(\tilde{z}_j) > 0$ respectively for $j \in \mathcal{I}$ and that $p_j = q_j = \infty$ for $j \notin \mathcal{I}$.

Bound on $\mathrm{E}((\cdot)^+)$

In the absence of full distribution, it would be meaningless to evaluate the optimal objective as depicted in the stochastic optimization Problem (3). Instead, we aim to minimize a good upper bound on the objective function. Such approach of soliciting inventory decisions based on partial demand information is not new. In the 50s, Scarf [35] considered a min-max newsvendor problem with uncertain demand $\tilde{d}$ given by only its mean and standard deviations. Scarf was able to obtain solutions to the tight upper bound of the newsvendor problem. The central idea in addressing such problem is to solicit a good upper bound on $\mathrm{E}((\cdot)^+)$, which appears at the objective of the newsvendor problem and also in Problem (3). The following result is well known.

Proposition 2 (Optimal upper bound, Lo [29] and Bertsimas and Popescu [14]) *Let $\tilde{z}$ be a random variable in $[-\mu, \infty)$ with mean μ and standard deviation σ, then for all $a \geq -\mu$,*

$$\mathrm{E}((\tilde{z} - a)^+) \leq \begin{cases} \frac{1}{2}\left(-a + \sqrt{\sigma^2 + a^2}\right) & \text{if } a \geq \frac{\sigma^2 - \mu^2}{2\mu} \\ -a\frac{\mu^2}{\mu^2 + \sigma^2} + \mu\frac{\sigma^2}{\mu^2 + \sigma^2} & \text{if } a < \frac{\sigma^2 - \mu^2}{2\mu} \end{cases}.$$

Moreover, the bound is tight.

Interestingly, Bertsimas and Thiele [13] used the bound of Proposition 2 to calibrate the "budget of uncertainty" parameter (see Bertsimas and Sim [12]) in their robust inventory models. Unfortunately, it is generally computationally intractable to evaluate tight probability bounds involving multivariate random variables with known moments and support information (see Bertsimas and Popescus [14]). We adopt the bounds of Chen and Sim [19] to evaluate the expected positive of an affine sum of random variables under the Assumption U.

Definition 2 *We say a function, $f(\boldsymbol{z})$ is* non-zero crossing *with respect to $\boldsymbol{z} \in \mathcal{W}$ if at least one of the following conditions hold*

1. $f(\boldsymbol{z}) \geq 0 \qquad \forall \boldsymbol{z} \in \mathcal{W}$

2. $f(\boldsymbol{z}) \leq 0 \qquad \forall \boldsymbol{z} \in \mathcal{W}$.

Theorem 2 *(Chen and Sim [19]) Let $\tilde{\boldsymbol{z}} \in \Re^N$ be a multivariate random variable under the Assumption U. Then*

$$\mathrm{E}((y_0 + \boldsymbol{y}'\tilde{\boldsymbol{z}})^+) \leq \pi(y_0, \boldsymbol{y}),$$

where $\pi(y_0, \boldsymbol{y})$ is given by

$$\begin{aligned}
\pi(y_0, \boldsymbol{y}) = \min \quad & r_1 + r_2 + r_3 + r_4 + r_5 \\
\text{s.t.} \quad & y_{10} + \sum_{j:\bar{z}_j<\infty} s_j \bar{z}_j + \sum_{j:\underline{z}_j<\infty} t_j \underline{z}_j \leq r_1 \\
& s_j = 0 \ \forall j : \bar{z}_j = \infty, \qquad t_j = 0 \ \forall j : \underline{z}_j = \infty \\
& 0 \leq r_1 \\
& \boldsymbol{s} - \boldsymbol{t} = \boldsymbol{y}_1 \\
& \boldsymbol{s}, \boldsymbol{t} \geq 0 \\
& \sum_{j:\bar{z}_j<\infty} d_j \bar{z}_j + \sum_{j:\underline{z}_j<\infty} h_j \underline{z}_j \leq r_2 \\
& d_j = 0 \ \forall j : \bar{z}_j = \infty, \qquad h_j = 0 \ \forall j : \underline{z}_j = \infty \\
& y_{20} \leq r_2 \\
& \boldsymbol{d} - \boldsymbol{h} = -\boldsymbol{y}_2 \\
& \boldsymbol{d}, \boldsymbol{h} \geq 0 \\
& \tfrac{1}{2} y_{30} + \tfrac{1}{2} \|(y_{30}, \boldsymbol{\Sigma}^{1/2} \boldsymbol{y}_3)\|_2 \leq r_3 \\
& \inf_{\mu>0} \tfrac{\mu}{e} \exp\left(\tfrac{y_{40}}{\mu} + \tfrac{\|\boldsymbol{u}\|_2^2}{2\mu^2}\right) \leq r_4 \\
& u_j \geq p_j y_{4j} \ \forall j : p_j < \infty, \qquad y_{4j} \leq 0 \ \forall j : p_j = \infty \\
& u_j \geq -q_j y_{4j} \ \forall j : q_j < \infty, \qquad y_{4j} \geq 0 \ \forall j : q_j = \infty \\
& y_{50} + \inf_{\mu>0} \tfrac{\mu}{e} \exp\left(-\tfrac{y_{50}}{\mu} + \tfrac{\|\boldsymbol{v}\|_2^2}{2\mu^2}\right) \leq r_5 \\
& v_j \geq q_j y_{5j} \ \forall j : q_j < \infty, \qquad y_{5j} \geq 0 \ \forall j : q_j = \infty \\
& v_j \geq -p_j y_{5j} \ \forall j : p_j < \infty, \qquad y_{5j} \leq 0 \ \forall j : p_j = \infty \\
& y_{10} + y_{20} + y_{30} + y_{40} + y_{50} = y_0 \\
& \boldsymbol{y}_1 + \boldsymbol{y}_2 + \boldsymbol{y}_3 + \boldsymbol{y}_4 + \boldsymbol{y}_5 = \boldsymbol{y}. \\
& r_i, y_{i0} \in \Re, \boldsymbol{y}_i, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{d}, \boldsymbol{h} \in \Re^N, i = 1, \ldots, 5, \boldsymbol{u}, \boldsymbol{v} \in \Re^N
\end{aligned} \tag{6}$$

Moreover, the bound is tight if $y_0 + \boldsymbol{y}'\boldsymbol{z}$ is a non-zero crossing function with respect to $\boldsymbol{z} \in \mathcal{W}$. That is, if

$$y_0 + \boldsymbol{y}'\boldsymbol{z} \geq 0 \qquad \forall \boldsymbol{z} \in \mathcal{W}$$

we have $\mathrm{E}\left((\,y_0 + \boldsymbol{y}'\boldsymbol{z}\,)^+\right) = \pi(y_0, \boldsymbol{y}) = y_0$. *Likewise, if*

$$y_0 + \boldsymbol{y}'\boldsymbol{z} \le 0 \qquad \forall \boldsymbol{z} \in \mathcal{W},$$

we have $\mathrm{E}\left((\,y_0 + \boldsymbol{y}'\boldsymbol{z}\,)^+\right) = \pi(y_0, \boldsymbol{y}) = 0$.

Remark 1: Due to the presence of the constraints, $\inf_{\mu>0} \mu \exp\left(\frac{a}{\mu} + \frac{b^2}{\mu^2}\right) \le c$, the set of constraints in Problem (6) is not exactly second order cone representable (see Ben-Tal and Nemirovski [8]). Fortunately, using a few second order cones, we can accurately approximate such constraints to good a level of numerical precision. The interested readers can refer to Chen and Sim [19].

Remark 2: Note that under the Assumption U, it is not necessary to provide all the information such as forward and backward deviations and supports. So, whenever such information is unavailable, we can assign an infinite value to the corresponding parameter. For instance, suppose the factor $\tilde{z}_j$ has support in $[-\mu, \infty)$, standard deviation, σ and unknown forward and backward deviations, we would set $\underline{z}_j = \mu$, $\bar{z}_j = \infty$, $p_j = q_j = \infty$. With more information on the factors, the bound of Problem (6) is never worse off.

Remark 3: In the absence of uncertainty, the non-zero crossing condition ensures that the bound is tight. That is, $y^+ = \mathrm{E}(y^+) = \pi(y, \mathbf{0})$.

We next show for a univariate random variable with one-sided support, the bound of Theorem 2 is as tight as Proposition 2.

Proposition 3 *Let $\tilde{z}$ be a random variable in $[-\mu, \infty)$ with mean μ and standard deviation σ, then for all $a \ge -\mu$,*

$$\mathrm{E}((\tilde{z} - a)^+) \le \pi(-a, 1) = \begin{cases} \dfrac{1}{2}\left(-a + \sqrt{\sigma^2 + a^2}\right) & \textit{if } a \ge \frac{\sigma^2 - \mu^2}{2\mu} \\ -a\dfrac{\mu^2}{\mu^2 + \sigma^2} + \mu\dfrac{\sigma^2}{\mu^2 + \sigma^2} & \textit{if } a < \frac{\sigma^2 - \mu^2}{2\mu} \end{cases}$$

Proof : See Appendix A.

We can further improve the bound if the distribution of the random variable $\tilde{z}$ is sufficiently light tailed such that the forward and backward deviations are close to its standard deviations, such as those of normal and uniform distribution. Figure 1 compares the bounds of $\mathrm{E}((\tilde{z} - a)^+)$ in which $\mu = 1$ and $\sigma = \sigma_f(\tilde{z}) = \sigma_b(\tilde{z}) = 2$. Bound 1 corresponds to the bound of Proposition 2, while Bound 2 corresponds to the bound Theorem 2. Clearly, despite the lack of tightness results, incorporating the forward and backward deviations can potentially improve the bound on $\mathrm{E}((\tilde{z} - a)^+)$.

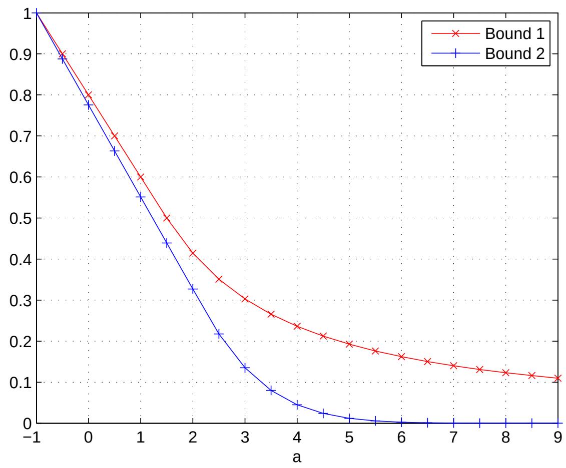

Figure 1: Comparing bounds of $\mathrm{E}((\tilde{z}-a)^+)$

3.1 Tractable Replenishment Policies

Having introduced the demand uncertainty model, a suitable approximation of the replenishment policy $x_t(\tilde{\boldsymbol{d}}_{t-1})$ is needed to obtain a tractable formulation. That is, we seek a formulation in which the policy can be obtained by solving a optimization problem that runs in polynomial time and scalable across time period. We review two tractable replenishment policies, static as well as linear with respect to the random factors of demand, which have recently been proposed in robust optimization literatures. We introduced a new replenishment policy, known as the truncated linear replenishment policy, which improves over these policies.

Static replenishment policy

The static replenishment policy proposed by Bertsimas and Thiele [13], is a tractable one where the order decisions are not influenced by the random factors of demand. A tractable model under such replenishment policy is as follows:

$$\begin{aligned} Z_{SRP} = \min \quad & \sum_{t=1}^{T}\Big(c_t x_t^0 + h_t\pi\left(y_{t+1}^0, \boldsymbol{y}_{t+1}\right) + b_t\pi\left(-y_{t+1}^0, -\boldsymbol{y}_{t+1}\right)\Big) \\ \text{s.t.} \quad & y_{t+1}^0 = y_t^0 + x_{t-L}^0 - d_t^0 \qquad t = 1,\dots,T \\ & y_{t+1}^k = y_t^k - d_t^k \qquad k = 1,\dots,N, t = 1,\dots,T \\ & 0 \le x_t^0 \le S_t \qquad t = 1,\dots,T-L, \end{aligned} \tag{7}$$

with y_1^0 being the initial inventory position and $y_1^k = 0$ for all $k = 1, \dots, N$. For $L \geq 1$, x_t^0 are the known committed orders made at time periods $t = 1 - L, \dots, 0$.

Theorem 3 *The expected cost of the stochastic inventory problem under the static replenishment policy,*

$$x_t^{SRP}(\tilde{\boldsymbol{d}}_{t-1}) = x_t^{0*} \qquad t = 1, \dots, T - L$$

in which x_t^{0*}, $t = 1, \dots, T - L$ are the optimal solution of Problem (7), is at most Z_{SRP}.

Proof : See Appendix B.

Linear replenishment policy

A more refined replenishment policy introduced in Ben Tal et. al [4], and Chen, Sim and Sun [17] is the linear replenishment policy where the order decisions are affinely dependent on the random factors of demand. That is

$$x_t^{LRP}(\tilde{\boldsymbol{d}}_{t-1}) = x_t^0 + \boldsymbol{x}_t'\tilde{\boldsymbol{z}},$$

in which the vector $\boldsymbol{x}_t = (x_t^1, \dots, x_t^N)$ satisfies the following non-anticipative constraints,

$$x_t^k = 0 \qquad \forall k \geq N_t.$$

Since the order decision is made at the beginning of the tth period, the non-anticipative constraints ensure that the linear replenishment policy is not influenced by demand factors that are unavailable up to the beginning of the tth period. The model for the linear replenishment policy is as follows:

$$\begin{array}{rll} Z_{LRP} = \min & \displaystyle\sum_{t=1}^{T} \Big(c_t x_t^0 + h_t \pi\left(y_{t+1}^0, \boldsymbol{y}_{t+1}\right) + b_t \pi\left(-y_{t+1}^0, -\boldsymbol{y}_{t+1}\right)\Big) & \\ \text{s.t.} & y_{t+1}^k = y_t^k + x_{t-L}^k - d_t^k \qquad k = 0, \dots, N, t = 1, \dots, T & \\ & x_t^k = 0 \qquad \forall k \geq N_t, t = 1, \dots, T - L & \\ & 0 \leq x_t^0 + \boldsymbol{x}_t'\boldsymbol{z} \leq S_t \qquad \forall \boldsymbol{z} \in \mathcal{W} \qquad t = 1, \dots, T - L, & \end{array} \tag{8}$$

with y_1^0 being the initial inventory position and $y_1^k = 0$ for all $k = 1, \dots, N$. For $L \geq 1$, x_t^0 are the known committed orders made at time periods $t = 1 - L, \dots, 0$.

Note that under the box uncertainty set $\mathcal{W}$, it is well known that the robust counterpart

$$0 \leq x_t^0 + \boldsymbol{x}_t'\boldsymbol{z} \leq S_t \qquad \forall \boldsymbol{z} \in \mathcal{W}$$

can be represented concisely as linear constraints. See for instance Ben-Tal et. al. [4] and Chen, Sim and Sun [17]. Therefore, Problem (8) is essentially a second order cone optimization problem.

Theorem 4 *The expected cost of the stochastic inventory problem under the linear replenishment policy,*

$$x_t^{LRP}(\tilde{\boldsymbol{d}}_{t-1}) = x_t^{0*} + \boldsymbol{x}_t^{*\prime}\tilde{\boldsymbol{z}} \qquad t = 1, \ldots, T-L$$

in which x_t^{k*}, $k = 0, \ldots, N, t = 1, \ldots, T-L$ *are the optimal solution of Problem (8), is at most* Z_{LRP}*. Moreover,* $Z_{LRP} \leq Z_{SRP}$*.*

Proof : See Appendix C.

Truncated linear decision policy

More recently, Chen, Sim, Sun and Zhang [18] developed a deflected linear decision rule that improves upon linear decision rule in the approximation of multi-period stochastic optimization problems. Specific to inventory control, we introduce the truncated linear replenishment policy as follows:

$$x_t^{TLRP}(\tilde{\boldsymbol{d}}_{t-1}) = \min\left\{\max\left\{x_t^0 + \boldsymbol{x}_t'\tilde{\boldsymbol{z}}, 0\right\}, S_t\right\},$$

in which the vector $\boldsymbol{x}_t = (x_t^1, \ldots, x_t^N)$ satisfies the following non-anticipative constraints,

$$x_t^k = 0 \qquad \forall k \geq N_t.$$

Note that the truncated linear replenishment policy has an embedded linear replenishment policy and that

$$0 \leq x_t^{TLRP}(\tilde{\boldsymbol{d}}_{t-1}) \leq S_t.$$

We first introduce the following result.

Theorem 5 *Let* $\tilde{\boldsymbol{z}} \in \Re^N$ *be a multivariate random variable under Assumption U. Then*

$$\mathrm{E}\left(\left(y^0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} + \sum_{i=1}^{p}\left(x_i^0 + \boldsymbol{x}_i{}'\tilde{\boldsymbol{z}}\right)^+\right)^+\right) \leq \eta((y^0, \boldsymbol{y}), (x_1^0, \boldsymbol{x}_1), \ldots, (x_p^0, \boldsymbol{x}_p)) \qquad (9)$$

where

$$\begin{aligned} & \eta((y^0, \boldsymbol{y}), (x_1^0, \boldsymbol{x}_1), \ldots, (x_p^0, \boldsymbol{x}_p)) \\ = & \min_{w_i^0, \boldsymbol{w}_i, i=1,\ldots,p}\left\{\pi\left(y^0 + \sum_{i=1}^{p} w_i^0, \boldsymbol{y} + \sum_{i=1}^{p}\boldsymbol{w}_i\right) + \sum_{i=1}^{p}\left(\pi(-w_i^0, -\boldsymbol{w}_i) + \pi(x_i^0 - w_i^0, \boldsymbol{x}_i - \boldsymbol{w}_i)\right)\right\}. \end{aligned}$$

Moreover, the bound is tight if $y^0 + \boldsymbol{y}'\boldsymbol{z} + \sum_{i=1}^{p}\left(x_i^0 + \boldsymbol{x}_i{}'\boldsymbol{z}\right)^+$ *and* $x_i^0 + \boldsymbol{x}_i{}'\boldsymbol{z}$*,* $i = 1, \ldots, p$ *are non-zero crossing functions with respect to* $\boldsymbol{z} \in \mathcal{W}$*.*

Proof : See Appendix D.

Remark: In the absence of uncertainty, the non-zero crossing condition ensures that the bound is tight. That is,

$$\left(y^0 + \sum_{i=1}^{p} \left(x_i^0\right)^+\right)^+ = \mathrm{E}\left(\left(y^0 + \sum_{i=1}^{p} \left(x_i^0\right)^+\right)^+\right) = \eta((y^0, \mathbf{0}), (x_1^0, \mathbf{0}), \ldots, (x_p^0, \mathbf{0})).$$

The model for the truncated linear replenishment policy is as follows:

$$\begin{aligned} Z_{TLRP} = \min \quad & \sum_{t=1}^{T} c_t \pi(x_t^0, \boldsymbol{x}_t) + \sum_{t=1}^{L}\Big(h_t \pi(y_{t+1}^0, \boldsymbol{y}_{t+1}) + b_t \pi(-y_{t+1}^0, -\boldsymbol{y}_{t+1})\Big) + \\ & \sum_{t=L+1}^{T}\Big(h_t \eta\left((y_{t+1}^0, \boldsymbol{y}_{t+1}), (-x_1^0, -\boldsymbol{x}_1), \ldots, (-x_{t-L}^0, -\boldsymbol{x}_{t-L})\right) + \\ & \qquad b_t \eta\left((-y_{t+1}^0, -\boldsymbol{y}_{t+1}), (x_1^0 - S_t, \boldsymbol{x}_1), \ldots, (x_{t-L}^0 - S_t, \boldsymbol{x}_{t-L})\right)\Big) \\ \text{s.t.} \quad & y_{t+1}^k = y_t^k + x_{t-L}^k - d_t^k \qquad k = 0, \ldots, N, t = 1, \ldots, T \\ & x_t^k = 0 \qquad \forall k \geq N_t, t = 1, \ldots, T-L \end{aligned} \tag{10}$$

with y_1^0 being the initial inventory position and $y_1^k = 0$ for all $k = 1, \ldots, N$. For $L \geq 1$, x_t^0 are the known committed orders made at time periods $t = 1-L, \ldots, 0$.

Theorem 6 *The expected cost of the stochastic inventory problem under the truncated linear replenishment policy,*

$$x_t^{TLRP}(\tilde{\boldsymbol{d}}_{t-1}) = \min\left\{\max\left\{x_t^{0*} + \boldsymbol{x}_t^{*\prime}\tilde{\boldsymbol{z}}, 0\right\}, S_t\right\} \qquad t = 1, \ldots, T-L$$

in which x_t^{k*}, $k = 0, \ldots, N$, $t = 1, \ldots, T-L$ *are the optimal solution of Problem (10), is at most* Z_{TLRP}. *Moreover,* $Z_{TLRP} \leq Z_{LDR}$.

Proof : See Appendix E.

Remark : For the case of unbounded ordering quantity, that is, $S_t = \infty$, the truncated linear replenishment policy becomes,

$$x_t^{TLRP}(\tilde{\boldsymbol{d}}_{t-1}) = \max\left\{x_t^0 + \boldsymbol{x}_t'\tilde{\boldsymbol{z}}, 0\right\},$$

and we can simplify Problem (10) as follows

$$\begin{aligned} Z_{TLRP} = \min \quad & \sum_{t=1}^{T} c_t \pi(x_t^0, \boldsymbol{x}_t) + \sum_{t=1}^{L} \Big(h_t \pi(y_{t+1}^0, \boldsymbol{y}_{t+1}) + b_t \pi(-y_{t+1}^0, -\boldsymbol{y}_{t+1}) \Big) + \\ & \quad \sum_{t=L+1}^{T} \Big(h_t \eta \left((y_{t+1}^0, \boldsymbol{y}_{t+1}), (-x_1^0, -\boldsymbol{x}_1), \ldots, (-x_{t-L}^0, -\boldsymbol{x}_{t-L}) \right) + \\ & \qquad b_t \pi \left(-y_{t+1}^0, -\boldsymbol{y}_{t+1} \right) \Big) \\ \text{s.t.} \quad & y_{t+1}^k = y_t^k + x_{t-L}^k - d_t^k \qquad k = 0, \ldots, N, t = 1, \ldots, T \\ & x_t^k = 0 \qquad \forall k \geq N_t, t = 1, \ldots, T - L. \end{aligned} \tag{11}$$

Rolling horizon implementation

In practice, the replenishment policies may be implemented in a rolling horizon manner. That is, the model is solved repeatedly with updated data. See Zipkin [40] and Ben-Tal et. al. [4]. In the rolling horizon implementation, new order decisions are generated whenever demand realizations are observed. The inventory manager re-solves in each period t Problem (10) for the remaining periods $t, t+1, \ldots T$, starting from initial inventory that are the actual ones, that is, those resulting from the earlier decisions and the demand realized in periods $1, 2, \ldots, t-1$. The implementation requires only minor adjustments to the model. The first run of any rolling horizon model is identical to the corresponding run of a fixed horizon model. Then, to execute the second run we reduce the horizon by one period, set the order decisions according to their optimal values in the first run, and fix the starting inventory according to the realization of demand and replenishment decision from the previous period. Since more accurate information are used each time the model is solved, the results will only improve.

4 Other extensions

In this section, we discuss some extension to the basic model.

4.1 Fixed ordering cost

Unfortunately, with fixed ordering cost, the inventory replenishment problem becomes nonconvex and is much harder to address. Using the idea of Bertsimas and Thiele [13], we can formulate a restricted problem where the time period in which the orders that can be placed is determined at the start of the planning horizon as follows:

$$\begin{aligned} Z_{STOCF} = \min \quad & \sum_{t=1}^{T}\left(\mathrm{E}\left(c_t x_t(\tilde{\boldsymbol{d}}_{t-1}) + K_t r_t + h_t(y_{t+1}(\tilde{\boldsymbol{d}}_t))^+\right) + \mathrm{E}\left(b_t(y_{t+1}(\tilde{\boldsymbol{d}}_t))^-\right)\right). \\ \text{s.t.} \quad & y_{t+1}(\tilde{\boldsymbol{d}}_t) = y_t(\tilde{\boldsymbol{d}}_{t-1}) + x_{t-L}(\tilde{\boldsymbol{d}}_{t-L-1}) - \tilde{d}_t \qquad t = 1,\ldots,T \\ & 0 \le x_t(\tilde{\boldsymbol{d}}_{t-1}) \le S_t r_t \qquad t = 1,\ldots,T-L \\ & r_t \in \{0,1\} \qquad\qquad t = 1,\ldots,T-L. \end{aligned} \tag{12}$$

In Problem (12), inventory can only be replenished at period where the corresponding binary variable r_t takes the value of one. We can then incorporate the tractable replenishment policies developed in the previous section. The resulting optimization model is a conic integer program, which is already addressed in commercial solvers such as CPLEX 10.1. Admittedly, algorithms for solving conic integer program are still at their infancy. On the theoretical front, Atamtürk and Narayanan [1] recently described general-purpose conic mixed-integer rounding cuts based on polyhedral conic substructures of second-order conic sets, which can be readily incorporated in branch-and-bound algorithms that solve continuous conic programming relaxations at the nodes of the search tree. Their preliminary computational experiments shows that the new cuts are quite effective in reducing the integrality gap of continuous relaxations of conic mixed-integer programs.

4.2 Supply chain networks

The models we have presented in the preceding section can also be extended to more complex supply chain networks such as the series system, or more generally the tree network. These are multi-stage system where goods transit from one stage to the next stage, each time moving closer to their final destination. In many supply chains, the main storage hubs, or the sources of the network, receive their supplies from outside manufacturing plants in a tree-like hierarchical structure and send items throughout the network until they finally reach the stores, or the sinks of the network. The extension to tree structure uses the concept of echelon inventory and closely follows Bertsimas and Thiele [13]. We refer interested readers to their paper.

5 Computation Studies

In our computation studies, we compared the policy obtained from deterministic approximation of robust optimization model to policies obtained from stochastic dynamic programming. We consider a multi-period inventory control problem evaluated under the truncated linear decision policy framework. The demand process we considered is motivated by Graves [27] as follows:

$$d_t(\tilde{\boldsymbol{z}}) = \tilde{z}_t + \alpha\tilde{z}_{t-1} + \alpha\tilde{z}_{t-2} + \ldots + \alpha\tilde{z}_1 + \mu, \tag{13}$$

where the mean $\mu = 100$ and shocks factors $\tilde{z}_t$ being independently uniform distributed random variables in $[-20, 20]$, which has standard, forward and backward deviations of approximately 11.55. To reduce the search effort needed for obtaining solutions from the dynamic programming algorithm, we have used the uniform distributions which has bounded support. Observe that the demand process of Equation (13) for $t \geq 2$ can be expressed recursively as

$$d_t(\tilde{\boldsymbol{z}}) = d_{t-1}(\tilde{\boldsymbol{z}}) - (1-\alpha)\tilde{z}_{t-1} + \tilde{z}_t. \tag{14}$$

Hence, this demand process is an integrated moving average (IMA) process of order $(0, 1, 1)$. See also Box et al. [16]. Note that given $\bar{\mu} = d_{t-1}(\tilde{\boldsymbol{z}}) - (1-\alpha)\tilde{z}_{t-1}$ at time period t, the distribution of $d_t(\tilde{\boldsymbol{z}})$ is uniform in $[-20 + \bar{\mu}, 20 + \bar{\mu}]$. A range of demand process can be modeled by varying α. With $\alpha = 0$, the demand process follows an i.i.d process of uniformly distributed random variables in $[80, 120]$. As α grows, the demand process becomes non-stationary and less stable with increasing variance. When $\alpha = 1$, the demand process is random walk on a continuous state space.

We consider problem with $T = 5$ so that the demand, $d_T(\tilde{\boldsymbol{z}})$ is nonnegative for all $\alpha \in [0, 1]$. The lead-time, L is zero, unit ordering cost $c_t = 2$, and unit holding cost $h_t = 7$ at all periods $t = 1, \ldots, T$. The backlog costs at periods $t = 1, \ldots, T-1$, are $b_t = 10$. Since unfulfilled demands are lost at the end of T, we set a relatively high backlog cost, $b_T = 500$, to penalize unfulfilled demand at the last period. The maximum order quantities, S_t, are set to 140 and we vary α from 0 to 1 in steps of 0.2.

We obtained the truncated linear replenishment policy (TLRP) by formulating Problem (10) using an in-house developed software, *PROF* (Platform for Robust Optimization Formulation). The Matlab based software is essentially a modeling platform for robust optimization that contains reusable functions for modeling multi-period robust optimization using decision rules. After formulating the model, it calls upon CPLEX 10.1 for solution. We have implemented bounds for $\pi(\cdot)$ of Equation (6) and $\eta(\cdot)$ of Theorem 5. For $T = 5$, the size of the problem we consider is presented in Table 1. Our computation was carried out on a 2.8GHz desktop with 2Mb memory. Computational time depends on the number of period. For $T = 5$, it typically took less than 1 seconds to solve for the TLRP model. For $T = 15$, it typically took about 4 seconds.

Number of affine constraints	5911
Number of free variables	3366
Number of non-negative variables	1700
Number of $\mathcal{L}^2$ cones	12
Number of $\mathcal{L}^3$ cones	1226
Number of $\mathcal{L}^4$ cones	38
Number of $\mathcal{L}^5$ cones	50
Number of $\mathcal{L}^6$ cones	72
Number of $\mathcal{L}^7$ cones	27

Table 1: Size of the TLRP model, where $\mathcal{L}^n = \{(x_0, \boldsymbol{x}) \in \Re \times \Re^{n-1} \; : \; \|\boldsymbol{x}\|_2 \leq x_0\}$.

t	z_t	d_t	x_t^{TRLP}	y_{t+1}
1	18.0	118.0	102.5	-15.5
2	19.3	126.5	136.3	-5.7
3	19.8	134.7	138.5	-1.9
4	-14.2	108.6	140.0	29.5
5	-2.0	115.2	105.2	19.5

Table 2: A sample path of the truncated linear replenishment policy.

The below example, solved for $\alpha = 0.4$, serves as an illustration of the truncated linear replenishment policy.

$$\begin{bmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \\ x_4^0 \\ x_5^0 \end{bmatrix} = \begin{bmatrix} 102.5 \\ 103.0 \\ 101.4 \\ 107.5 \\ 105.6 \end{bmatrix} \qquad \begin{bmatrix} \boldsymbol{x}_1' \\ \boldsymbol{x}_2' \\ \boldsymbol{x}_3' \\ \boldsymbol{x}_4' \\ \boldsymbol{x}_5' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1.85 & 0 & 0 & 0 \\ 0.24 & 1.70 & 0 & 0 \\ 0.19 & 0.18 & 1.42 & 0 \\ 0.32 & 0.32 & 0.38 & 1.40 \end{bmatrix}$$

In Table (2), we show a sample path of the truncated linear replenishment policy,

$$x_i^{TLRP}(\boldsymbol{z}) = \min\{(x_i^0 + \boldsymbol{x}_i'\boldsymbol{z})^+, 140\}.$$

We benchmark our solutions against heuristics based on dynamic programming, where the optimal replenishment policy can be characterized by the following backward recursion.

$$\begin{aligned} J_t(y_t, d_{t-1}, z_{t-1}) &= \min_{0 \le x \le S_t} \mathrm{E}\Big(c_t x + r_t(y_t + x - d_t(d_{t-1}, z_{t-1}, \tilde{z}_t)) + \\ &\qquad J_{t+1}(y_t + x - d_t(d_{t-1}, z_{t-1}, \tilde{z}_t), d_t(d_{t-1}, z_{t-1}, \tilde{z}_t), \tilde{z}_t) \Big) \end{aligned}$$

where $d_t(d_{t-1}, z_{t-1}, \tilde{z}_t) = d_{t-1} - (1-\alpha)z_{t-1} + \tilde{z}_t$ and $r_t(u) = h_t \max(u, 0) + b_t \max(-u, 0)$. The optimal replenishment policy at time t is a function of the current inventory position y_t, previous demand d_{t-1} and shock, z_{t-1} as follows.

$$\begin{aligned} x_t^{OPT}(y_t, d_{t-1}, z_{t-1}) &= \arg\min_{0 \le x \le S_t} \mathrm{E}\Big(c_t x + r_t(y_t + x - d_t(d_{t-1}, z_{t-1}, \tilde{z}_t)) + \\ &\qquad J_{t+1}(y_t + x - d_t(d_{t-1}, z_{t-1}, \tilde{z}_t), d_t(d_{t-1}, z_{t-1}, \tilde{z}_t), \tilde{z}_t) \Big). \end{aligned}$$

Due to the large state spaces, we consider two heuristics. The first being a state independent base-stock policy (BSP), where we compute the replenishment policy recursively by ignoring the dependency of previous demands and shocks as follows,

$$J_t^{BSP}(y_t) = \min_{0 \le x \le S_t} \mathrm{E}\Big(c_t x + r_t(y_t + x - d_t(d_{t-1}, z_{t-1}, \tilde{z}_t)) + J_{t+1}^{BSP}(y_t + x - d_t(d_{t-1}, z_{t-1}, \tilde{z}_t) \Big),$$

The replenishment policy is given by

$$x_t^{BSP}(y_t) = \arg\min_{0 \le x \le S_t} \mathrm{E}\Big(c_t x + r_t(y_t + x - d_t(d_{t-1}, z_{t-1}, \tilde{z}_t)) + J_{t+1}^{BSP}(y_t + x - d_t(d_{t-1}, z_{t-1}, \tilde{z}_t) \Big).$$

Under capacity limit on order quantities, the modified base-stock policy is optimal when the demands are independently distributed, which occurs only when $\alpha = 0$. See Federgruen and Zipkin [22]. Note that in practice, we are unable to compute expectations of functions exactly. Instead, at every dynamic programming recursion, we computed the value functions approximately using sampling approximations from $1,000$ instances of demand realizations. In another level of approximation, we discretized the state space to integer values so that the value functions are finite vectors instead of continuous functions. The other heuristic we considered is an adaptive myopic policy (MP), where the replenishment level is derived by minimizing the following one-period expected cost as described below

$$x_t^{MP}(y_t, d_{t-1}, z_{t-1}) = \arg\min_{0 \le x \le S_t} \mathrm{E}\Big(c_t x + r_t(y_t + x - d_t(d_{t-1}, z_{t-1}\tilde{z}_t)) \Big).$$

In contrast with the optimal dynamic programming recursion, the adaptive myopic policy optimizes only the current period expected cost and ignores all subsequent costs.

After obtaining the policies, we compared them using $100,000$ simulated inventory runs and reported the sample mean and the sample error of mean over all the simulated runs. In Table (3) we use TLRP,

α	TLPR	BSP	MP	$\hat{\sigma}$(TLRP)	$\hat{\sigma}$(BSP)	$\hat{\sigma}$(MP)	BSP/TLPR	MP/TLPR
1	2416	3290	2760	5.5	17.1	14.5	1.36	1.14
0.8	2048	2573	2138	2.3	11.2	8.7	1.26	1.04
0.6	1716	2056	1784	1.0	6.1	4.7	1.20	1.04
0.4	1550	1769	1611	0.5	3.4	2.3	1.14	1.04
0.2	1515	1576	1539	0.5	1.2	0.9	1.04	1.02
0	1512	1513	1526	0.4	0.5	0.5	1.00	1.01

Table 3: Performance of truncated linear replenishment policy

BSP and MP to denote the sample mean of the expected cost under the simulated runs when the replenishment policies are the truncated linear replenishment policy, state-independent base-stock policy and adaptive myopic policy respectively. We denote the sample error of their mean by $\hat{\sigma}(\cdot)$. The last two columns, BSP/TLRP and MP/TLRP show the performance of the truncated linear replenishment policy against the state-independent base-stock policy and adaptive myopic policy respectively.

In our test cases, we solved the dynamic program using samples from the the exact distribution. Interestingly, for independent demands where state independent modified base-stock policy is optimal, the truncated linear replenishment policy solved using partial distribution information gives objective values comparable to the sample average approximation to stochastic dynamic programming. For highly correlated demands ($\alpha \geq 0.6$), TLPR outperforms BSP by more than 20%. It is never worst off against MP, and when $\alpha = 1$ TLRP outperforms by 14%. This is not surprising, as truncated linear replenishment policy is a non-myopic policy that adapts the order quantity to all past realizations of demand, whereas state independent base-stock policy uses only information from the most recent period to adjust the order quantity. While the state independent base-stock policy obtained from dynamic programming uses one control parameter, truncated linear decision policy has more control parameters.

6 Conclusions

In this paper we have proposed a tractable robust optimization methodology that uses past demand realization to adaptively control multi-period inventory that are subjected to demand uncertainty. Our formulation is developed using a factor-based demand model, which has the ability to incorporate business factors as well as forecast effects of trend, seasonality and cyclic variation. Our computational results clearly demonstrate that our robust optimization model can be used to analyze multi-period problems efficiently.

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A Proof of Proposition 3

Proof : The bound $\mathrm{E}((\tilde{z}-a)^+) \le \pi(-a,1)$ follows directly from Theorem 2. Since the bound of Proposition 2 is tight, it suffices to show

$$\pi(-a,1) \le \begin{cases} \frac{1}{2}\left(-a+\sqrt{\sigma^2+a^2}\right) & \text{if } a \ge \frac{\sigma^2-\mu^2}{2\mu} \\ -a\frac{\mu^2}{\mu^2+\sigma^2}+\mu\frac{\sigma^2}{\mu^2+\sigma^2} & \text{if } a < \frac{\sigma^2-\mu^2}{2\mu} \end{cases}$$

With $\underline{z}=\mu$ and $p=q=\bar{z}=\infty$, we first simplify the bound as follows:

$$\begin{aligned} \pi(y_0,\boldsymbol{y}) = \min \quad & r_1+r_2+r_3 \\ \text{s.t.} \quad & y_{10}+t_1\mu \le r_1 \\ & 0 \le r_1 \\ & -t_1 = y_{11} \\ & t_1 \ge 0 \\ & h_1\mu \le r_2 \\ & y_{20} \le r_2 \\ & h_1 = y_{21} \\ & h_1 \ge 0 \\ & \tfrac{1}{2}y_{30}+\tfrac{1}{2}\sqrt{y_{30}^2+\sigma^2 y_{31}^2} \le r_3 \\ & y_{10}+y_{20}+y_{30} = -a \\ & y_{11}+y_{21}+y_{31} = 1 \\ = \min \quad & (y_{10}-y_{11}\mu)^+ + \max\{y_{21}\mu, y_{20}\} + \tfrac{1}{2}y_{30}+\tfrac{1}{2}\sqrt{y_{30}^2+\sigma^2 y_{31}^2} \\ \text{s.t.} \quad & y_{11} \le 0 \\ & y_{21} \ge 0 \\ & y_{10}+y_{20}+y_{30} = -a \\ & y_{11}+y_{21}+y_{31} = 1. \end{aligned} \tag{15}$$

Clearly, with $y_{10} = y_{20} = 0$, $y_{30} = -a$, $y_{11} = y_{21} = 0$ and $y_{31} = 1$, we see that $\pi(y_0, \boldsymbol{y}) \leq -\frac{1}{2}a + \frac{1}{2}\sqrt{a^2+\sigma^2}$. Now for $a < \frac{\sigma^2-\mu^2}{2\mu}$, we let $y_{10} = y_{11} = 0$,

$$\begin{aligned} y_{20} &= \mu\tfrac{\sigma^2-\mu^2-2\mu a}{\mu^2+\sigma^2}, \\ y_{21} &= \tfrac{\sigma^2-\mu^2-2\mu a}{\mu^2+\sigma^2} \geq 0, \\ y_{30} &= (\mu+a)\tfrac{\mu^2-\sigma^2}{\mu^2+\sigma^2}, \\ y_{31} &= 2\mu\tfrac{\mu+a}{\mu^2+\sigma^2}. \end{aligned}$$

which are feasible in Problem (15). Hence,

$$\begin{aligned} \pi(-a,1) &\leq (y_{10}-y_{11}\mu)^+ + \max\{y_{21}\mu, y_{20}\} + \tfrac{1}{2}y_{30} + \tfrac{1}{2}\sqrt{y_{30}^2+\sigma^2 y_{31}^2} \\ &= -a - \frac{1}{2}(\mu+a)\frac{\mu^2-\sigma^2}{\mu^2+\sigma^2} + \frac{1}{2}\underbrace{\sqrt{(a+\mu)^2}}_{=a+\mu} \\ &= -a\frac{\mu^2}{\mu^2+\sigma^2} + \mu\frac{\sigma^2}{\mu^2+\sigma^2}. \end{aligned}$$

■

B Proof of Theorem 3

Proof : Under the static replenishment policy and using the factor-based demand model, the inventory position at the end of period t is given by

$$\begin{aligned} y_{t+1}^{SRP}(\tilde{\boldsymbol{d}}_t) &= y_1^0 + \sum_{\tau=1}^{\min\{L,t\}} x_{\tau-L}^0 + \sum_{\tau=L+1}^{t} x_{\tau-L}^{SRP}(\tilde{\boldsymbol{d}}_{\tau-L-1}) - \sum_{\tau=1}^{t} d_\tau(\tilde{\boldsymbol{z}}) \\ &= y_1^0 + \sum_{\tau=1}^{\min\{L,t\}} x_{\tau-L}^0 + \sum_{\tau=L+1}^{t} x_{\tau-L}^{0*} - \sum_{\tau=1}^{t} d_\tau^0 - \sum_{\tau=1}^{t}\sum_{k=1}^{N} d_\tau^k \tilde{z}_k \\ &= \underbrace{y_1^0 + \sum_{\tau=1}^{\min\{L,t\}} x_{\tau-L}^0 + \sum_{\tau=L+1}^{t} x_{\tau-L}^{0*} - \sum_{\tau=1}^{t} d_\tau^0}_{=y_{t+1}^{0*}} + \sum_{k=1}^{N} \underbrace{\left(\sum_{\tau=1}^{t}(-d_\tau^k)\right)}_{=y_{t+1}^{k*}} \tilde{z}_k \\ &= y_{t+1}^{0*} + \sum_{\tau=1}^{N} y_{t+1}^{k*}\tilde{z}_k \end{aligned}$$

where y_{t+1}^{k*} $k = 0, \ldots, N$, $t = 1, \ldots, T$ are the optimum solutions of Problem (7). Clearly, the static replenishment policy, $x_t^{SRP}(\tilde{\boldsymbol{d}}_{t-1})$ is feasible in Problem (3). Moreover, by Theorem 2, we have

$$\begin{aligned} & \mathrm{E}\left(c_t x_t^{SRP}(\tilde{\boldsymbol{d}}_{t-1}) + h_t\left(y_{t+1}^{SRP}(\tilde{\boldsymbol{d}}_t)\right)^+ + b_t\left(y_{t+1}^{SRP}(\tilde{\boldsymbol{d}}_t)\right)^-\right) \\ =\ & \mathrm{E}\left(c_t x_t^{0*} + h_t\left(y_{t+1}^{0*} + \sum_{k=1}^{N} y_{t+1}^{k*}\tilde{z}_k\right)^+ + b_t\left(-y_{t+1}^{0*} - \sum_{k=1}^{N} y_{t+1}^{k*}\tilde{z}_k\right)^+\right) \qquad (16) \\ \leq\ & c_t x_t^{0*} + h_t\pi\left(y_{t+1}^{0*}, \boldsymbol{y}_{t+1}^*\right) + b_t\pi\left(-y_{t+1}^{0*}, -\boldsymbol{y}_{t+1}^*\right). \end{aligned}$$

Hence, $Z_{STOC} \leq Z_{SRP}$. ■

C Proof of Theorem 4

Proof : Observe that Problem (8) with additional constraints $x_t^k = 0$, $k = 1, \ldots, N$, $t = 1 \ldots, T - L$ gives the same feasible constraint set as Problem (7). Moreover, the objective functions of both problems are the same. Hence, $Z_{LRP} \leq Z_{SRP}$. Under the linear replenishment policy, the inventory position at the end of period t is given by

$$\begin{aligned}
y_{t+1}^{LRP}(\tilde{\boldsymbol{d}}_t) &= y_1^0 + \sum_{\tau=1}^{\min\{L,t\}} x_{\tau-L}^0 + \sum_{\tau=L+1}^{t} x_{\tau-L}^{LRP}(\tilde{\boldsymbol{d}}_{\tau-L-1}) - \sum_{\tau=1}^{t} d_\tau(\tilde{\boldsymbol{z}}) \\
&= y_1^0 + \sum_{\tau=1}^{\min\{L,t\}} x_{\tau-L}^0 + \sum_{\tau=L+1}^{t} \left(x_{\tau-L}^{0*} + \sum_{k=1}^{N} x_{\tau-L}^{k*} \tilde{z}_k \right) - \sum_{\tau=1}^{t} d_\tau^0 - \sum_{\tau=1}^{t} \sum_{k=1}^{N} d_\tau^k \tilde{z}_k \\
&= \underbrace{y_1^0 + \sum_{\tau=1}^{\min\{L,t\}} x_{\tau-L}^0 + \sum_{\tau=L+1}^{t} x_{\tau-L}^{0*} - \sum_{\tau=1}^{t} d_\tau^0}_{=y_{t+1}^{0*}} + \sum_{k=1}^{N} \underbrace{\left(\sum_{\tau=1}^{t} (x_{\tau-L}^{k*} - d_\tau^k) \right)}_{=y_{t+1}^{k*}} \tilde{z}_k \\
&= y_{t+1}^{0*} + \sum_{\tau=1}^{N} y_{t+1}^{k*} \tilde{z}_k
\end{aligned}$$

where y_{t+1}^{k*} $k = 0, \ldots, N$, $t = 1, \ldots, T$ are the optimum solutions of Problem (8). Clearly, the linear replenishment policy, $x_t^{LRP}(\tilde{\boldsymbol{d}}_{t-1})$ is feasible in Problem (3). Moreover, by Theorem 2 and that $\tilde{\boldsymbol{z}}$ being zero mean random variables, we have

$$\begin{aligned}
& \mathrm{E}\left(c_t x_t^{LRP}(\tilde{\boldsymbol{d}}_{t-1}) + h_t \left(y_{t+1}^{LRP}(\tilde{\boldsymbol{d}}_t) \right)^+ + b_t \left(y_{t+1}^{LRP}(\tilde{\boldsymbol{d}}_t) \right)^- \right) \\
= \;& \mathrm{E}\left(c_t \left(x_t^{0*} + \boldsymbol{x}_t^{*\prime} \tilde{\boldsymbol{z}} \right) + h_t \left(y_{t+1}^{0*} + \sum_{k=1}^{N} y_{t+1}^{k*} \tilde{z}_k \right)^+ + b_t \left(-y_{t+1}^{0*} - \sum_{k=1}^{N} y_{t+1}^{k*} \tilde{z}_k \right)^+ \right) \\
\leq \;& c_t x_t^{0*} + h_t \pi \left(y_{t+1}^{0*}, \boldsymbol{y}_{t+1}^* \right) + b_t \pi \left(-y_{t+1}^{0*}, -\boldsymbol{y}_{t+1}^* \right).
\end{aligned} \tag{17}$$

Hence, $Z_{STOC} \leq Z_{LRP}$. ■

D Proof of Theorem 5

Proof : We first show the following bound

$$\left(y + \sum_{i=1}^{p} x_i^+ \right)^+ \leq \left(y + \sum_{i=1}^{p} w_i \right)^+ + \sum_{i=1}^{p} \left((-w_i)^+ + (x_i - w_i)^+ \right) \tag{18}$$

for all w_i, $i = 1, \ldots, p$. Note that for any scalars a, b

$$a^+ + b^+ \geq (a+b)^+ \tag{19}$$

$$a^+ + b^+ = a^+ + (b^+)^+ \geq (a + b^+)^+. \tag{20}$$

Therefore, we have

$$\begin{array}{rll} & \left(y+\sum_{i=1}^{p} w_i\right)^{+} + \sum_{i=1}^{p}\left((-w_i)^{+} + (x_i - w_i)^{+}\right) & \\ \geq & \left(y+\sum_{i=1}^{p}\left(w_i + (-w_i)^{+} + (x_i - w_i)^{+}\right)\right)^{+} & \text{from Inequality (20)} \\ = & \left(y+\sum_{i=1}^{p}\left(w_i^{+} + (x_i - w_i)^{+}\right)\right)^{+} & \\ \geq & \left(y+\sum_{i=1}^{p} x_i^{+}\right)^{+} & \text{from Inequality (19).} \end{array}$$

For notational convenience, we denote $y(\tilde{\boldsymbol{z}}) = y^0 + \boldsymbol{y}'\tilde{\boldsymbol{z}}$, $x_i(\tilde{\boldsymbol{z}}) = x_i^0 + \boldsymbol{x}_i{}'\tilde{\boldsymbol{z}}$ and $w_i(\tilde{\boldsymbol{z}}) = w_i^0 + \boldsymbol{w}_i{}'\tilde{\boldsymbol{z}}$. To prove Inequality (9), it suffices to show that for any $w_i^0, \boldsymbol{w}_i, i = 1, \ldots, p$, we have

$$\begin{array}{rl} & \pi\left(y^0 + \sum_{i=1}^{p} w_i^0, \boldsymbol{y} + \sum_{i=1}^{p} \boldsymbol{w}_i\right) + \sum_{i=1}^{p}\left(\pi(-w_i^0, -\boldsymbol{w}_i) + \pi(x_i^0 - w_i^0, \boldsymbol{x}_i - \boldsymbol{w}_i)\right) \\ \geq & \mathrm{E}\left(\left(y(\tilde{\boldsymbol{z}}) + \sum_{i=1}^{p} w_i(\tilde{\boldsymbol{z}})\right)^{+}\right) + \sum_{i=1}^{p}\left(\mathrm{E}\left((-w_i(\tilde{\boldsymbol{z}}))^{+}\right) + \mathrm{E}\left((x_i(\tilde{\boldsymbol{z}}) - w_i(\tilde{\boldsymbol{z}}))^{+}\right)\right) \\ \geq & \mathrm{E}\left(\left(y(\tilde{\boldsymbol{z}}) + \sum_{i=1}^{p} x_i(\tilde{\boldsymbol{z}})^{+}\right)^{+}\right), \end{array}$$

where the first inequality follows from Theorem 2 and the last inequality follows from Inequality (18).

To prove the tightness of the bound, we consider the case when $x_i^0 + \boldsymbol{x}_i{}'\boldsymbol{z}$, $i = 1, \ldots, p$ are non-zero crossing functions with respect to $\boldsymbol{z} \in \mathcal{W}$. Let

$$\mathcal{K} = \{k \ : \ x_k^0 + \boldsymbol{x}_k{}'\boldsymbol{z} \geq 0 \ \forall \boldsymbol{z} \in \mathcal{W}\}.$$

Hence,

$$y^0 + \boldsymbol{y}'\boldsymbol{z} + \sum_{i=1}^{p}\left(x_i^0 + \boldsymbol{x}_i{}'\boldsymbol{z}\right)^{+} = y^0 + \boldsymbol{y}'\boldsymbol{z} + \sum_{i\in\mathcal{K}}\left(x_i^0 + \boldsymbol{x}_i{}'\boldsymbol{z}\right) \qquad \forall \boldsymbol{z} \in \mathcal{W}.$$

Therefore, if

$$y^0 + \boldsymbol{y}'\boldsymbol{z} + \sum_{i=1}^{p}\left(x_i^0 + \boldsymbol{x}_i{}'\boldsymbol{z}\right)^{+} \geq 0 \qquad \forall \boldsymbol{z} \in \mathcal{W}$$

or equivalently,

$$y^0 + \boldsymbol{y}'\boldsymbol{z} + \sum_{i\in\mathcal{K}}\left(x_i^0 + \boldsymbol{x}_i{}'\boldsymbol{z}\right) \geq 0 \qquad \forall \boldsymbol{z} \in \mathcal{W},$$

we have

$$\begin{array}{rl} & \mathrm{E}\left(\left(y^0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} + \sum_{i=1}^{p}\left(x_i^0 + \boldsymbol{x}_i{}'\tilde{\boldsymbol{z}}\right)^{+}\right)^{+}\right) \\ = & \mathrm{E}\left(\left(y^0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} + \sum_{i\in\mathcal{K}}\left(x_i^0 + \boldsymbol{x}_i{}'\tilde{\boldsymbol{z}}\right)\right)^{+}\right) \\ = & y^0 + \sum_{i\in\mathcal{K}} x_i^0. \end{array}$$

Likewise, if

$$y^0 + \boldsymbol{y}'\boldsymbol{z} + \sum_{i=1}^{p} \left(x_i^0 + \boldsymbol{x}_i'\boldsymbol{z}\right)^+ \leq 0 \qquad \forall \boldsymbol{z} \in \mathcal{W}$$

or equivalently,

$$y^0 + \boldsymbol{y}'\boldsymbol{z} + \sum_{i\in\mathcal{K}} \left(x_i^0 + \boldsymbol{x}_i'\boldsymbol{z}\right) \leq 0 \qquad \forall \boldsymbol{z} \in \mathcal{W},$$

we have

$$\mathrm{E}\left(\left(y^0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} + \sum_{i=1}^{p} \left(x_i^0 + \boldsymbol{x}_i'\tilde{\boldsymbol{z}}\right)^+\right)^+\right) = \mathrm{E}\left(\left(y^0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} + \sum_{i\in\mathcal{K}} \left(x_i^0 + \boldsymbol{x}_i'\tilde{\boldsymbol{z}}\right)\right)^+\right) = 0.$$

Indeed, for all $k \in \mathcal{K}$, let $(w_i^0, \boldsymbol{w}_i) = (x_i^0, \boldsymbol{x}_i)$ and for all $k \notin \mathcal{K}$, $(w_i^0, \boldsymbol{w}_i) = (0, \boldsymbol{0})$. Therefore, using the tightness result of Theorem 2, we have

$$\begin{aligned}
& \mathrm{E}\left(\left(y^0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} + \sum_{i=1}^{p} \left(x_i^0 + \boldsymbol{x}_i'\tilde{\boldsymbol{z}}\right)^+\right)^+\right) \\
\leq\ & \eta((y^0, \boldsymbol{y}), (x_1^0, \boldsymbol{x}_1), \ldots, (x_p^0, \boldsymbol{x}_p)) \\
=\ & \min_{w_i^0, \boldsymbol{w}_i, i=1,\ldots,p} \left\{ \pi\left(y^0 + \sum_{i=1}^{p} w_i^0, \boldsymbol{y} + \sum_{i=1}^{p} \boldsymbol{w}_i\right) + \sum_{i=1}^{p} \left(\pi(-w_i^0, -\boldsymbol{w}_i) + \pi(x_i^0 - w_i^0, \boldsymbol{x}_i - \boldsymbol{w}_i)\right) \right\} \\
\leq\ & \pi\left(y^0 + \sum_{i\in\mathcal{K}} x_i^0, \boldsymbol{y} + \sum_{i\in\mathcal{K}} \boldsymbol{x}_i\right) + \sum_{i\in\mathcal{K}} \left(\underbrace{\pi(-x_i^0, -\boldsymbol{x}_i)}_{=0} + \pi(0, \boldsymbol{0})\right) + \\
& \quad \sum_{i\notin\mathcal{K}} \left(\pi(-0, -\boldsymbol{0}) + \underbrace{\pi(x_i^0, \boldsymbol{x}_i)}_{=0}\right) \\
=\ & \pi\left(y^0 + \sum_{i\in\mathcal{K}} x_i^0, \boldsymbol{y} + \sum_{i\in\mathcal{K}} \boldsymbol{x}_i\right) \\
=\ & \begin{cases} y^0 + \sum_{i\in\mathcal{K}} x_i^0 & \text{if } y^0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} + \sum_{i\in\mathcal{K}} \left(x_i^0 + \boldsymbol{x}_i'\tilde{\boldsymbol{z}}\right) \geq 0 \ \forall \boldsymbol{z} \in \mathcal{W} \\ 0 & \text{if } y^0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} + \sum_{i\in\mathcal{K}} \left(x_i^0 + \boldsymbol{x}_i'\tilde{\boldsymbol{z}}\right) \leq 0 \ \forall \boldsymbol{z} \in \mathcal{W} \end{cases} \\
=\ & \mathrm{E}\left(\left(y^0 + \boldsymbol{y}'\tilde{\boldsymbol{z}} + \sum_{i=1}^{p} \left(x_i^0 + \boldsymbol{x}_i'\tilde{\boldsymbol{z}}\right)^+\right)^+\right).
\end{aligned}$$

■

E Proof of Theorem 6

Proof : We first show that $Z_{TLRP} \leq Z_{LDR}$. Let $x_t^{k\dagger}$, $k = 0, \ldots, N$, $t = 1, \ldots, T - L$ and $y_{t+1}^{k\dagger}$, $k = 0, \ldots, N$, $t = 1, \ldots, T$ be the optimal solution to Problem (8), which is also feasible in Problem

(10). Based on the following inequality,

$$\begin{array}{rl} & \eta((y^0,\boldsymbol{y}),(x_1^0,\boldsymbol{x}_1),\ldots,(x_p^0,\boldsymbol{x}_p)) \\ = & \min\limits_{w_i^0,\boldsymbol{w}_i,i=1,\ldots,p} \left\{\pi\left(y^0+\sum_{i=1}^p w_i^0,\boldsymbol{y}+\sum_{i=1}^p \boldsymbol{w}_i\right)+\sum_{i=1}^p\Big(\pi(-w_i^0,-\boldsymbol{w}_i)+\pi(x_i^0-w_i^0,\boldsymbol{x}_i-\boldsymbol{w}_i)\Big)\right\} \\ \le & \pi\Big(y^0,\boldsymbol{y}\Big)+\sum_{i=1}^p \pi(x_i^0,\boldsymbol{x}_i), \end{array} \quad (21)$$

we have

$$\begin{array}{rl} Z_{TLRP} \le & \sum_{t=1}^T c_t\pi(x_t^{0\dagger},\boldsymbol{x}_t^\dagger)+\sum_{t=1}^L\Big(h_t\pi(y_{t+1}^{0\dagger},\boldsymbol{y}_{t+1}^\dagger)+b_t\pi(-y_{t+1}^{0\dagger},-\boldsymbol{y}_{t+1}^\dagger)\Big)+ \\ & \sum_{t=L+1}^T\Big(h_t\eta\Big((y_{t+1}^{0\dagger},\boldsymbol{y}_{t+1}^\dagger),(-x_1^{0\dagger},-\boldsymbol{x}_1),\ldots,(-x_{t-L}^{0\dagger},-\boldsymbol{x}_{t-L}^\dagger)\Big)+ \\ & \quad b_t\eta\Big((-y_{t+1}^{0\dagger},-\boldsymbol{y}_{t+1}^\dagger),(x_1^{0\dagger}-S_t,\boldsymbol{x}_1^\dagger),\ldots,(x_{t-L}^{0\dagger}-S_t,\boldsymbol{x}_{t-L}^\dagger)\Big)\Big) \\ \le & \sum_{t=1}^T c_t\pi(x_t^{0\dagger},\boldsymbol{x}_t^\dagger)+\sum_{t=1}^L\Big(h_t\pi(y_{t+1}^{0\dagger},\boldsymbol{y}_{t+1}^\dagger)+b_t\pi(-y_{t+1}^{0\dagger},-\boldsymbol{y}_{t+1}^\dagger)\Big)+ \\ & \sum_{t=L+1}^T\Big(h_t\pi\Big(y_{t+1}^{0\dagger},\boldsymbol{y}^\dagger\Big)+h_t\sum_{i=1}^{t-L}\pi(-x_i^{0\dagger},-\boldsymbol{x}_i^\dagger) \quad + \\ & b_t\pi\Big(-y_{t+1}^{0\dagger},-\boldsymbol{y}_{t+1}^\dagger\Big)+b_t\sum_{i=1}^{t-L}\pi(-x_i^{0\dagger}-S_t,\boldsymbol{x}_i^\dagger)\Big). \end{array}$$

Observe that since $x_t^{0\dagger}+\boldsymbol{x}_t^{\dagger\prime}\boldsymbol{z}\ge 0$, $-x_t^{0\dagger}-\boldsymbol{x}_t^{\dagger\prime}\boldsymbol{z}\le 0$ and $x_t^{0\dagger}-S_t+\boldsymbol{x}_t^{\dagger\prime}\boldsymbol{z}\le 0$ for all $\boldsymbol{z}\in\mathcal{W}$, we have from Theorem 2, $\pi(x_i^{0\dagger},\boldsymbol{x}_i^\dagger)=x_i^{0\dagger}$,$\pi(-x_i^{0\dagger},-\boldsymbol{x}_i^\dagger)=0$ and $\pi(x_i^{0\dagger}-S_t,\boldsymbol{x}_i^\dagger)=0$ for all $i=1,\ldots,T-L$. Hence,

$$Z_{TLRP}\le\sum_{t=1}^T\Big(c_tx_t^{0\dagger}+h_t\pi\Big(y_{t+1}^{0\dagger},\boldsymbol{y}^\dagger\Big)+b_t\pi\Big(-y_{t+1}^{0\dagger},-\boldsymbol{y}_{t+1}^\dagger\Big)\Big)=Z_{LRP}.$$

We next show that $Z_{STOC}\le Z_{TLRP}$. Under the truncated linear replenishment policy, the inventory position at the end of period t is given by

$$y_{t+1}^{TLRP}(\tilde{\boldsymbol{d}}_t)=y_1^0+\sum_{\tau=1}^{\min\{L,t\}}x_{\tau-L}^0+\sum_{\tau=L+1}^t x_{\tau-L}^{TLRP}(\tilde{\boldsymbol{d}}_{\tau-L-1})-\sum_{\tau=1}^t d_\tau(\tilde{\boldsymbol{z}}).$$

Let x_t^{k*}, $k=0,\ldots,N$, $t=1,\ldots,T-L$ and y_{t+1}^{k*}, $k=0,\ldots,N$, $t=1,\ldots,T$ be the optimal solution to Problem (10). It suffices to show that the following bounds

(a)

$$\mathrm{E}\Big(x_t^{TLRP}(\tilde{\boldsymbol{d}}_{t-1})\Big)\le\pi(x_t^{0*},\boldsymbol{x}_t^*).$$

(b) For $t = 1, \ldots, L$,

$$\mathrm{E}\left(\left(y_{t+1}^{TLRP}(\tilde{\boldsymbol{d}}_t)\right)^+\right) \leq \pi\left(y_{t+1}^{0*}, \boldsymbol{y}_{t+1}^*\right)$$

and

$$\mathrm{E}\left(\left(y_{t+1}^{TLRP}(\tilde{\boldsymbol{d}}_t)\right)^-\right) \leq \pi\left(-y_{t+1}^{0*}, -\boldsymbol{y}_{t+1}^*\right)$$

(c) For $t = L+1, \ldots, T$,

$$\mathrm{E}\left(\left(y_{t+1}^{TLRP}(\tilde{\boldsymbol{d}}_t)\right)^+\right) \leq \eta\left((y_{t+1}^{0*}, \boldsymbol{y}_{t+1}^*), (-x_1^{0*}, -\boldsymbol{x}_1^*), \ldots, (-x_{t-L}^{0*}, -\boldsymbol{x}_{t-L}^*)\right)$$

and

$$\mathrm{E}\left(\left(y_{t+1}^{TLRP}(\tilde{\boldsymbol{d}}_t)\right)^-\right) \leq \eta\left((-y_{t+1}^{0*}, -\boldsymbol{y}_{t+1}^*), (x_1^{0*} - S_t, \boldsymbol{x}_1^*), \ldots, (x_{t-L}^{0*} - S_t, \boldsymbol{x}_{t-L}^*)\right).$$

For Bound (a), we note that

$$\begin{aligned}
\mathrm{E}\left(x_t^{TLRP}(\tilde{\boldsymbol{d}}_{t-1})\right) &= \mathrm{E}\left(\min\left\{\max\left\{x_t^{0*} + \boldsymbol{x}_t^{*\prime}\tilde{\boldsymbol{z}}, 0\right\}, S_t\right\}\right) \\
&\leq \mathrm{E}\left(\max\left\{x_t^{0*} + \boldsymbol{x}_t^{*\prime}\tilde{\boldsymbol{z}}, 0\right\}\right) \\
&= \mathrm{E}\left(\left(x_t^{0*} + \boldsymbol{x}_t^{*\prime}\tilde{\boldsymbol{z}}\right)^+\right) \\
&\leq \pi(x_t^{0*}, \boldsymbol{x}_t^*).
\end{aligned}$$

We focus on deriving Bound (c), as the exposition of Bounds (b) is similar. Indeed, using the bound of

Theorem 5, we have for $t \geq L+1$,

$$
\begin{aligned}
& \mathrm{E}\left(\left(y_{t+1}^{TLRP}(\tilde{\boldsymbol{d}}_t)\right)^+\right) \\
= \; & \mathrm{E}\left(\left(y_1^0 + \sum_{\tau=1}^{L} x_{\tau-L}^0 + \sum_{\tau=L+1}^{t} x_{\tau-L}^{TLRP}(\tilde{\boldsymbol{d}}_{\tau-L-1}) - \sum_{\tau=1}^{t} d_\tau(\tilde{\boldsymbol{z}})\right)^+\right) \\
= \; & \mathrm{E}\left(\left(y_1^0 + \sum_{\tau=1}^{L} x_{\tau-L}^0 + \sum_{\tau=L+1}^{t} \min\left\{\max\left\{x_{\tau-L}^{0*} + {\boldsymbol{x}_{\tau-L}^*}'\tilde{\boldsymbol{z}}, 0\right\}, S_t\right\} - \sum_{\tau=1}^{t} d_\tau^0 - \sum_{\tau=1}^{t}\sum_{k=1}^{N} d_\tau^k \tilde{z}_k\right)^+\right) \\
\leq \; & \mathrm{E}\left(\left(y_1^0 + \sum_{\tau=1}^{L} x_{\tau-L}^0 + \sum_{\tau=L+1}^{t} \max\left\{x_{\tau-L}^{0*} + {\boldsymbol{x}_{\tau-L}^*}'\tilde{\boldsymbol{z}}, 0\right\} - \sum_{\tau=1}^{t} d_\tau^0 - \sum_{k=1}^{N}\sum_{\tau=1}^{t} d_\tau^k \tilde{z}_k\right)^+\right) \\
= \; & \mathrm{E}\Bigg(\Bigg(y_1^0 + \sum_{\tau=1}^{L} x_{\tau-L}^0 + \sum_{\tau=L+1}^{t} \left(x_{\tau-L}^{0*} + {\boldsymbol{x}_{\tau-L}^*}'\tilde{\boldsymbol{z}}\right) + \\
& \qquad \sum_{\tau=L+1}^{t} \max\left\{-x_{\tau-L}^{0*} - {\boldsymbol{x}_{\tau-L}^*}'\tilde{\boldsymbol{z}}, 0\right\} - \sum_{\tau=1}^{t} d_\tau^0 - \sum_{k=1}^{N}\sum_{\tau=1}^{t} d_\tau^k \tilde{z}_k\Bigg)^+\Bigg) \\
= \; & \mathrm{E}\Bigg(\Bigg(\underbrace{y_1^0 + \sum_{\tau=1}^{t} x_{\tau-L}^0 + \sum_{\tau=L+1}^{t} x_{\tau-L}^{0*} - \sum_{\tau=1}^{t} d_\tau^0}_{=y_{t+1}^{0*}} + \\
& \qquad \sum_{\tau=L+1}^{t} \left(-x_{\tau-L}^{0*} - {\boldsymbol{x}_{\tau-L}^*}'\tilde{\boldsymbol{z}}\right)^+ + \sum_{k=1}^{N} \underbrace{\left(\sum_{\tau=1}^{t}(x_{\tau-L}^k - d_\tau^k)\right)}_{=y_{t+1}^{k*}} \tilde{z}_k\Bigg)\Bigg) \\
= \; & \mathrm{E}\left(\left(y_{t+1}^{0*} + {\boldsymbol{y}_{t+1}^*}'\tilde{\boldsymbol{z}} + \sum_{\tau=L+1}^{t} \left(-x_{\tau-L}^{0*} - {\boldsymbol{x}_{\tau-L}^*}'\tilde{\boldsymbol{z}}\right)^+\right)^+\right) \\
\leq \; & \eta\left((y_{t+1}^{0*}, \boldsymbol{y}_{t+1}^*), (-x_1^{0*}, -\boldsymbol{x}_1^*), \ldots, (-x_{t-L}^{0*}, -\boldsymbol{x}_{t-L}^*)\right).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
& \mathrm{E}\left(\left(y_{t+1}^{TLRP}(\tilde{\boldsymbol{d}}_t)\right)^-\right) \\
= \ & \mathrm{E}\left(\left(-y_{t+1}^{TLRP}(\tilde{\boldsymbol{d}}_t)\right)^+\right) \\
= \ & \mathrm{E}\left(\left(-y_1^0 - \sum_{\tau=1}^{L} x_{\tau-L}^0 - \sum_{\tau=L+1}^{t} x_{\tau-L}^{TLRP}(\tilde{\boldsymbol{d}}_{\tau-L-1}) + \sum_{\tau=1}^{t} d_\tau(\tilde{\boldsymbol{z}})\right)^+\right) \\
= \ & \mathrm{E}\left(\left(-y_1^0 - \sum_{\tau=1}^{L} x_{\tau-L}^0 - \sum_{\tau=L+1}^{t} \min\left\{\max\left\{x_{\tau-L}^{0*} + \boldsymbol{x}_{\tau-L}^{*}{}'\tilde{\boldsymbol{z}}, 0\right\}, S_t\right\} - \sum_{\tau=1}^{t} d_\tau^0 + \sum_{\tau=1}^{t}\sum_{k=1}^{N} d_\tau^k \tilde{z}_k\right)^+\right) \\
\leq \ & \mathrm{E}\left(\left(-y_1^0 - \sum_{\tau=1}^{L} x_{\tau-L}^0 - \sum_{\tau=L+1}^{t} \min\left\{x_{\tau-L}^{0*} + \boldsymbol{x}_{\tau-L}^{*}{}'\tilde{\boldsymbol{z}}, S_t\right\} + \sum_{\tau=1}^{t} d_\tau^0 - \sum_{k=1}^{N}\sum_{\tau=1}^{t} d_\tau^k \tilde{z}_k\right)^+\right) \\
= \ & \mathrm{E}\Bigg(\Bigg(-y_1^0 - \sum_{\tau=1}^{L} x_{\tau-L}^0 - \sum_{\tau=L+1}^{t} \left(x_{\tau-L}^{0*} - \boldsymbol{x}_{\tau-L}^{*}{}'\tilde{\boldsymbol{z}}\right) + \\
& \qquad \sum_{\tau=L+1}^{t} \left(-\min\left\{S_t - x_{\tau-L}^{0*} - \boldsymbol{x}_{\tau-L}^{*}{}'\tilde{\boldsymbol{z}}, 0\right\}\right) + \sum_{\tau=1}^{t} d_\tau^0 + \sum_{k=1}^{N}\sum_{\tau=1}^{t} d_\tau^k \tilde{z}_k\Bigg)^+\Bigg) \\
= \ & \mathrm{E}\Bigg(\Bigg(\underbrace{-y_1^0 - \sum_{\tau=1}^{L} x_{\tau-L}^0 - \sum_{\tau=L+1}^{t} x_{\tau-L}^{0*} + \sum_{\tau=1}^{t} d_\tau^0}_{=-y_{t+1}^{0*}} + \\
& \qquad \sum_{\tau=L+1}^{t} \left(x_{\tau-L}^{0*} - S_t + \boldsymbol{x}_{\tau-L}^{*}{}'\tilde{\boldsymbol{z}}\right)^+ + \sum_{k=1}^{N} \underbrace{\left(\sum_{\tau=1}^{t}(-x_{\tau-L}^{k*} + d_\tau^k)\right)}_{=-y_{t+1}^{k*}} \tilde{z}_k\Bigg)\Bigg) \\
= \ & \mathrm{E}\left(\left(-y_{t+1}^{0*} - \boldsymbol{y}_{t+1}^{*}{}'\tilde{\boldsymbol{z}} + \sum_{\tau=L+1}^{t} \left(x_{\tau-L}^{0*} - S_t + \boldsymbol{x}_{\tau-L}^{*}{}'\tilde{\boldsymbol{z}}\right)^+\right)^+\right) \\
\leq \ & \eta\left((-y_{t+1}^{0*}, -\boldsymbol{y}_{t+1}^{*}), (x_1^{0*} - S_t, \boldsymbol{x}_1^{*}), \ldots, (x_{t-L}^{0*} - S_t, \boldsymbol{x}_{t-L}^{*})\right).
\end{aligned}
$$

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