Queueing Theory

The study of queueing theory requires some background in probability theory and in mathematical simulation.

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http://www.fd.cvut.cz/department/k611/PEDAGOG/K611THO.html

References

- Sanjay K. Bosse: An Introduction to Queueing Systems, Kluwer Academic, 2002
- Ivo Adan and Jacques Resing: Queueing theory, Eindhoven, The Netherlands,, p. 180, 2001 (free pdf, chapter 1-5)
- Andreas Willig: A Short Introduction to Queueing Theory, Berlin, p. 42, 1999 (free pdf)

Some situations in which queueing is important

Supermarket.
How long do customers have to wait at the checkouts? What happens with the waiting time during peak-hours? Are there enough checkouts?
Post office
In a post office there are counters specialized in e.g. stamps, packages, nancial transactions,
Are there enough counters? Separate queues or one common queue in front of counters with the same specialization?
Data communication.
In computer communication networks standard packages called cells are transmitted over links from one switch to the next. In each switch incoming cells can be buered when the incoming demand exceeds the link capacity. Once the buffer is full incoming cells will be lost.What is the cell delay at the switches? What is the fraction of cells that will be lost? What
is a good size of the buffer?

Utilization X Waiting Time

Queueing is a common phenomenon in our daily lives. It is impossible to avoid queueing as long as the number of people arrived is greater than the capacity of the service facility.

A long waiting time may result dissatisfaction among customers. However, if one tries to reduce the waiting time, he has to increase the investment. There is a trade off. How to balance efficiency and cost? At this time, Queueing Theory is introduced to solve the problems.

Queueing Theory is important when we study scheduling and network performance.

Queuing psychology: Can waiting in line be fun

Invironment - Occupied time feels shorter than unoccupied time

Expectation - Uncertain waits are longer than known, finite waits
Unexplained waits are longer than explained waits

Fair Play - Unfair waits are longer than equitable waits

Historie

Agner Krarup Erlang, a Danish engineer who worked for the Copenhagen Telephone Exchange, published the first paper on queueing theory in 1909.

David G. Kendall introduced an A/B/C queueing notation in

- 1953 Kendall notation
- 1969 Little's formula
- 1986 1st No - The Journal of Queueing Systems
- 1995 1st international symposium THO

Investigation Methods - Simulation

Advantages

- Simulation allows great flexibility in modeling complex systems, so simulation models can be highly valid
- Easy to compare alternatives
- Control experimental conditions

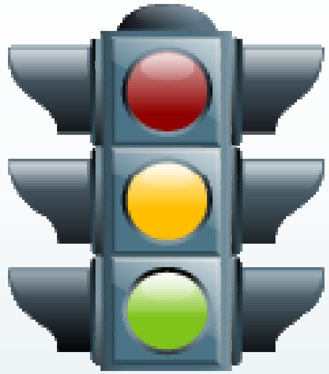
Disadvantages

- Stochastic simulations produce only estimates – with noise
- Simulations usually produce large volumes of output – need to summarize, statistically analyze appropriately

Estimation of quantities:

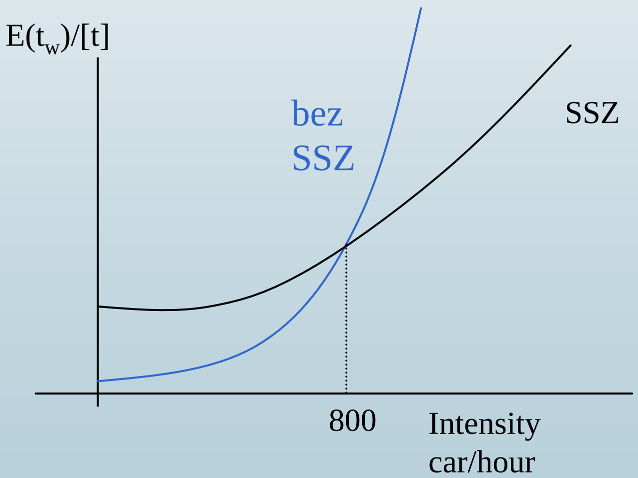
Expected value - Mean,
Probability - rel.frequency

Example - Traffic system

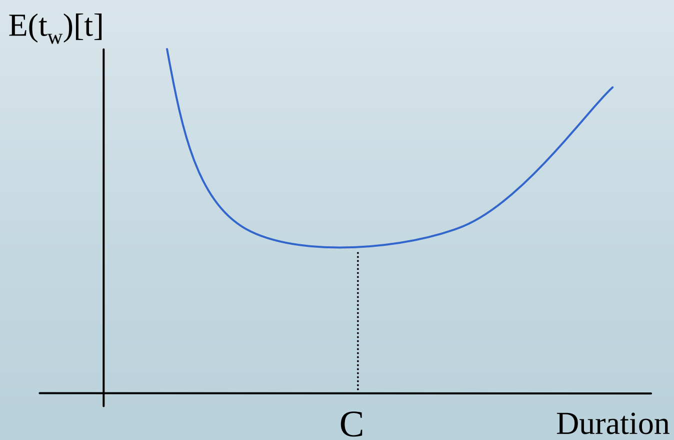


- Intersection Signal Control Optimization
- Dynamic signal control for congested traffic

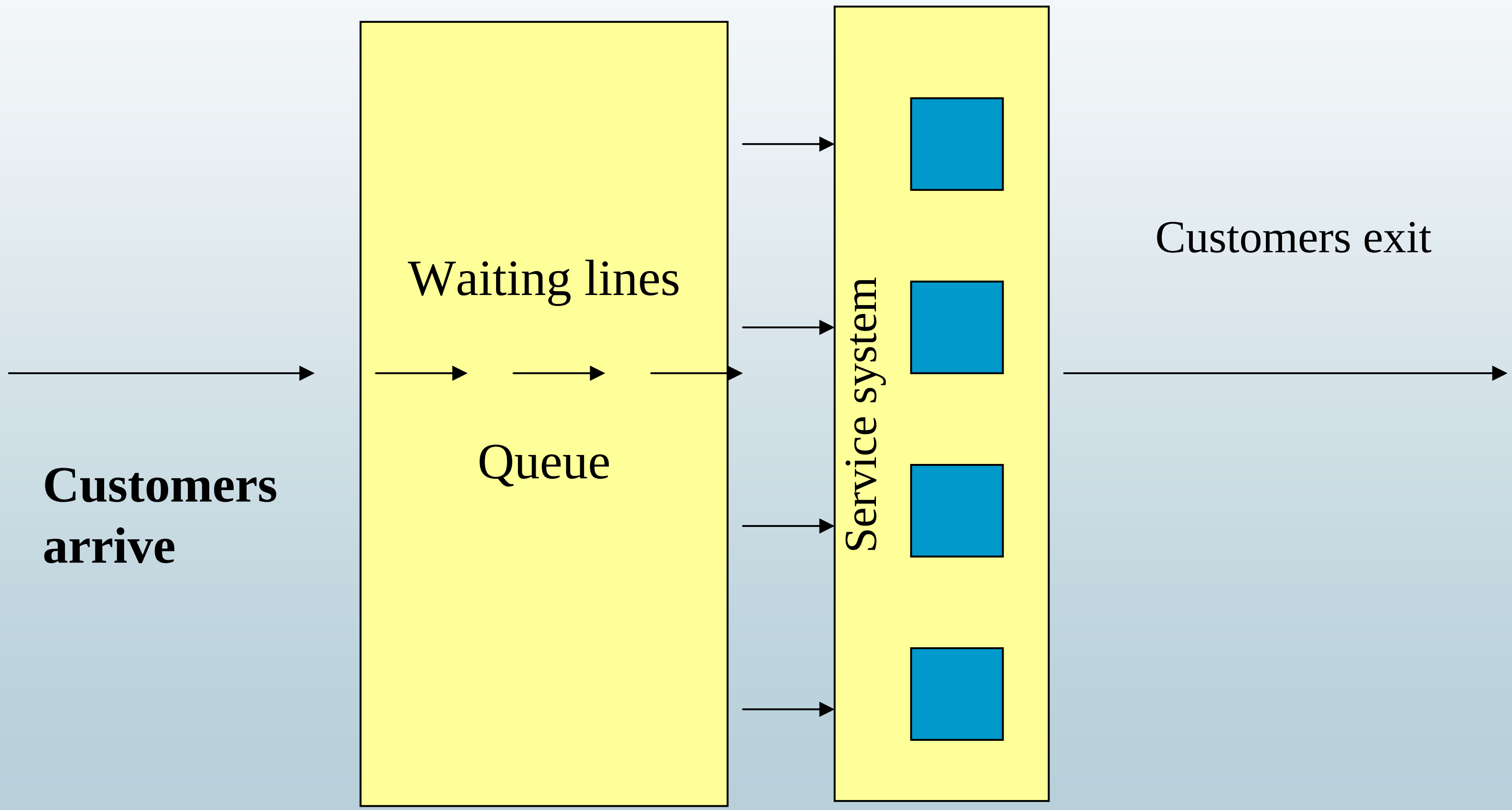
Dependance of delay and traffic intensity



Dependance of delay and duration of the cycle of signál plan



Queueing System



Fundamentals of probability

The set of all possible outcomes of random experiment - sample space

Event – subset of the sample space

Impossible event – empty set

All sample points – certain event

$$P(A) = \lim_{N \to \infty} \frac{N(A)}{N}$$

Property of probability

$$0 \le P(A) \le 1$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A') = 1 - P(A)$$

$$A \subset B \qquad P(A) \le P(B)$$

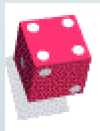
Independent Events

Two events are said to be independent if the result of the second event is not affected by the result of the first event

Example

$$P(A \cap B) = P(A).P(B)$$

A coin is tossed and a single 6-sided die is rolled. Find the probability of landing on the head side of the coin and rolling a 3 on the die.



$$P(\text{head}) = \frac{1}{2}$$

$$P(3) = \frac{1}{6}$$

$$P(\text{head and } 3) = P(\text{head}) \cdot P(3)$$

$$= \frac{1}{2} \cdot \frac{1}{6}$$

$$= \frac{1}{12}$$

Random variable

Discrete (can be realized with finite or countable set)

Continuous (can be realized with any of a range of values I⊂R)

– Cumulative distribution function

$$F(x) = P(X \le x)$$

- Probability density function

$$F(x) = \int_{-\infty}^{x} f(u)du \qquad F(b) - F(a) = \int_{a}^{b} f(u)du$$

Mean, Variance, Standart Deviation

Expected value

– Discrete

$$E[x] = \sum_i x_i P(X = x_i)$$

– Continuous

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

Properties:

E[c.X] = c.E[X]
E[X+Y] = E[X] + E[Y]
E[X .Y] = E[X] . E[Y] for independent X, Y

Variance

– Discrete

$$V[x] = \sum_i \left(x_i - E[X]\right)^2 P(X = x_i)$$

– Continuous

$$V[X] = \int_{-\infty}^{\infty} \left(x - E(X)\right)^2 \cdot f(x) dx$$

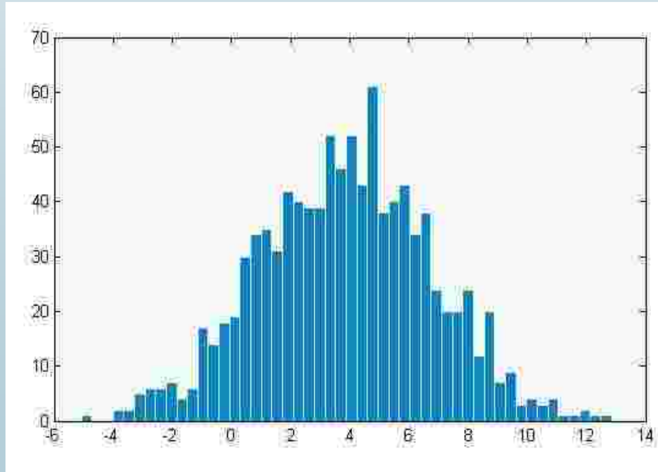
Probability Distributions

Discrete –if variable can take on finite (countable) number of values

- Binomial
- Poisson

Continuous – if variable can take on any value in interval

- Uniform
- Normal
- Exponential
- Erlang



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Binomial Distribution

probability of observing x successes in N trials, with the probability of success on a single trial denoted by p. The binomial distribution assumes that p is fixed for all trials.

$$p_i = P(X = i) = \binom{n}{i} p^i (1 - p)^{n-i}$$

$$E[X] = n \cdot p$$

$$V[X] = np(1 - p)$$

Příklad: Test consists of 10 questions, you can choose from 5 answers.

1. Probability, that all answers are wrong

$$P(X = 0) = \left(1 - \frac{1}{5}\right)^{10} \quad 0{,}11$$

2. Probability, that all answers are right

$$P(X = 0) = \left(\frac{1}{5}\right)^{10} \quad 0{,}0000001$$

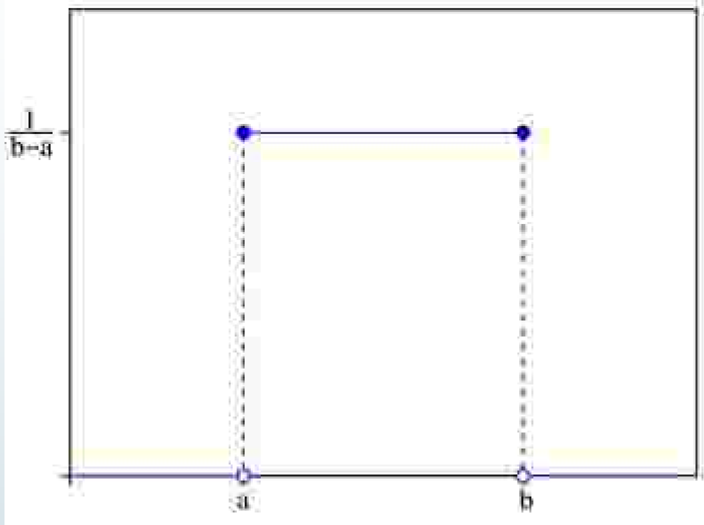
3. At least 7 correct answers

$$P(X \geq 7) = P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10) \quad 0{,}000864$$

4. Expected value of No of correct marked answers

$$E(X) = 10 \cdot \frac{1}{5} = 2$$

Uniform distribution



$$f(x) = \frac{1}{b-a}; \quad a \le x \le b$$

$$E[X] = \frac{a+b}{2}; \quad V[X] = \frac{(a+b)^2}{12}$$

Příklad: Subway go regularly every 5 minut. We come accidentally.

1. Probability,That we will wait at the most 1 minute

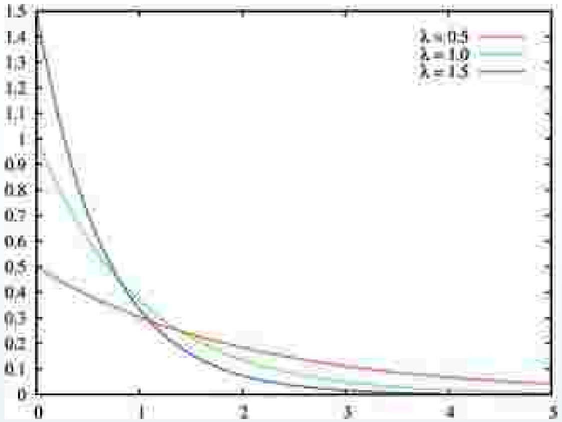
$$f(x) = \frac{1}{5}; \quad 0 \le x \le 5$$

$$P(x<1) = \int_0^1 \frac{1}{5} dx = \left. \frac{1}{5} x \right|_0^1 = \frac{1}{5}$$

2. Expected waiting time

$$E[X] = \frac{5}{2};$$

Exponential distribution



$$f(x)=\lambda e^{-\lambda x};\quad 0\le x\quad F(x)=1-e^{-\lambda x}$$

$$E[X]=\frac{1}{\lambda};\quad V[X]=\frac{1}{\lambda^2}$$

Example:

1. Period between two cars (out of town)Has exponential distribution λ. The mean number of cars per hour:

$$E[T]=\frac{1}{\lambda}$$

2. Trouble free time for new car ~ exp(1/10) [time unit - year].
   a) 5 year without any fault.

$$P\left(X\ge 5\right)=1-F\left(5\right)=1-\quad 1-e^{-\frac{1}{10}x}\quad =e^{-\frac{1}{2}}=0,606$$

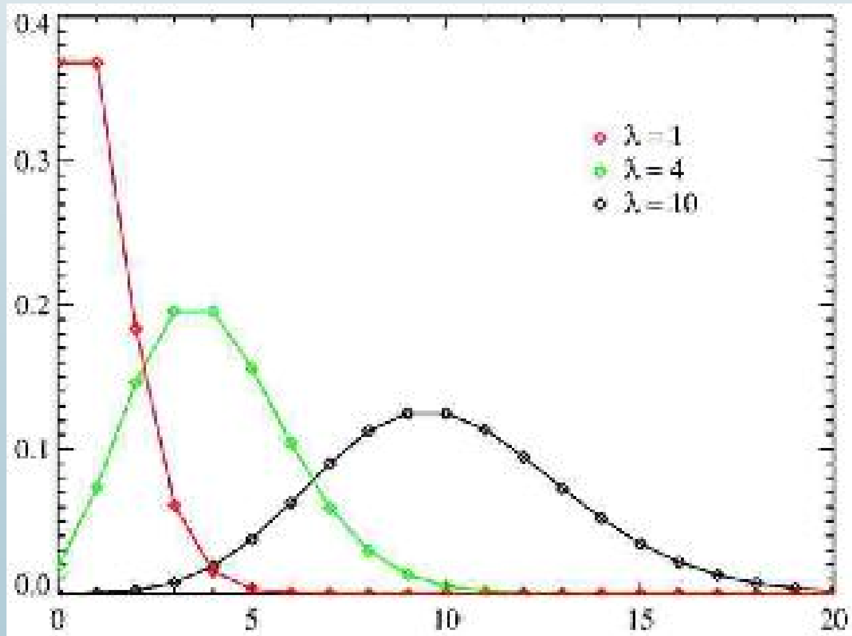
   b) Expected value for period without defect.

$$E[T]=\frac{1}{0.1}=10\ (let)$$

Poisson distribution of disrete random variable

probability of a number of events occurring in a fixed period of time if these events occur with a known average rate and independently of the time since the last event.

$$P(N(t)=k)=\frac{(\lambda t)^k}{k!}e^{-\lambda t} \qquad E[X]=\lambda\cdot t$$



Erlang distribution $X \sim Erlang(\lambda, k)$

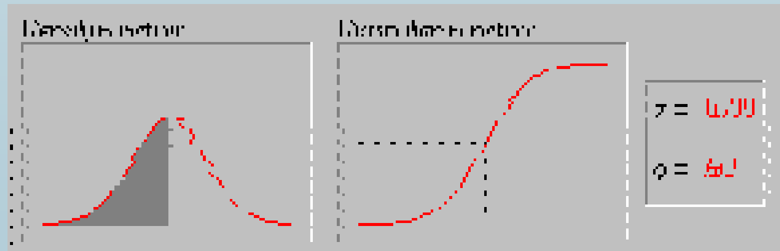
$$f(x) = \lambda e^{-\lambda x} \frac{(\lambda x)^{k-1}}{(k-1)!}; \quad 0 \le x \quad E[X] = \frac{k}{\lambda}; \quad V[X] = \frac{k}{\lambda^2}$$

Sum of k independent exponential random variable $X_i \sim \exp(\lambda)$

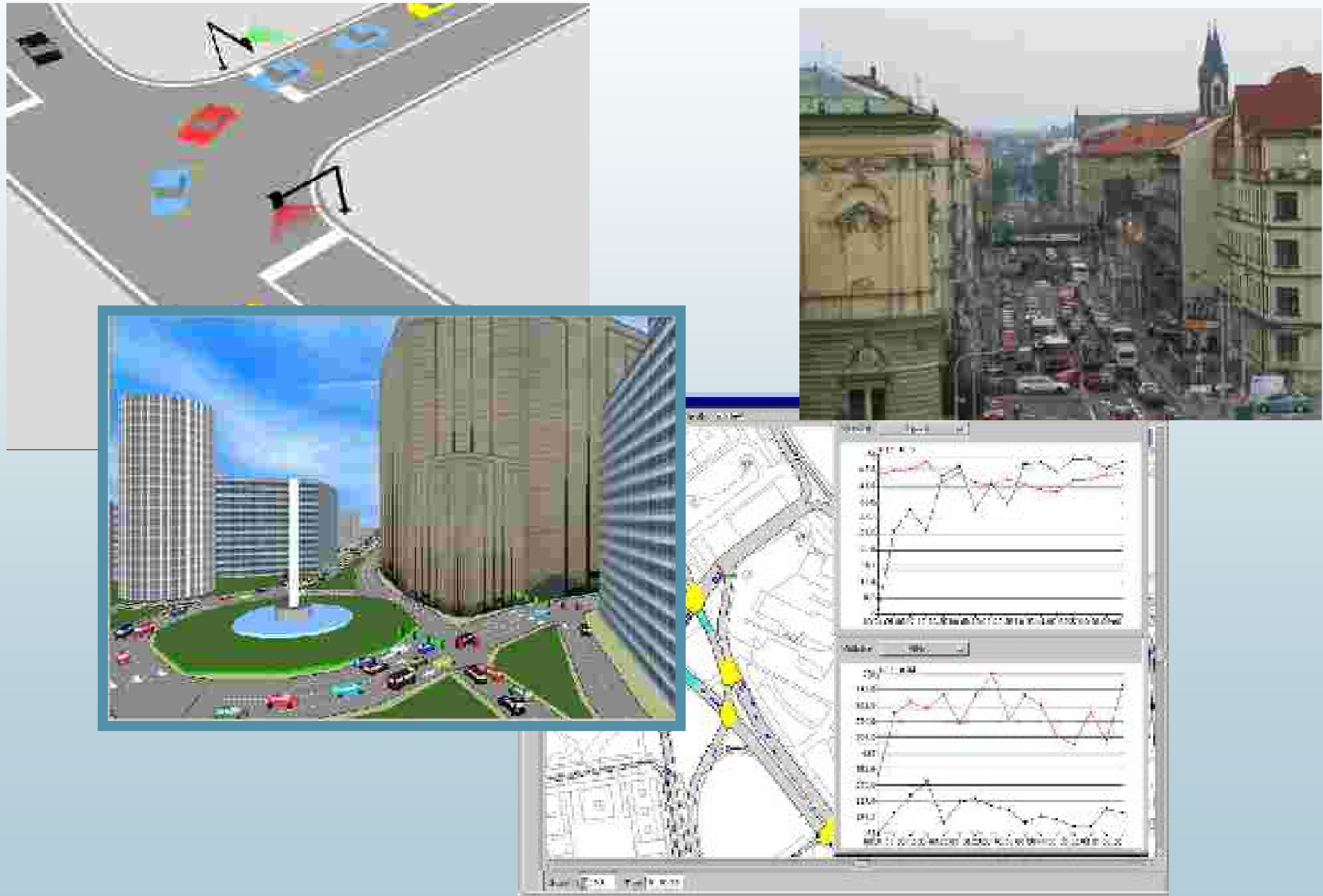
$X \sim Erlang(\lambda, k)$

Normal distribution $X \sim N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



Introduction to Simulation and Modeling of Queueing Systems



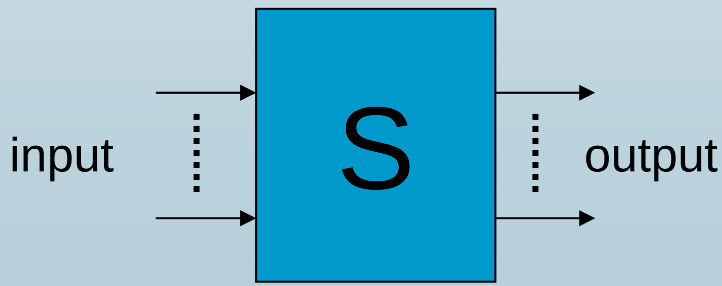
THE NATURE OF SIMULATION

System: A collection of entities (people, parts, messages, machines, servers, …) that act and interact together toward some end (Schmidt and Taylor, 1970)

To study system, **often make assumptions/approximations**, both logical and mathematical, about how it works

These assumptions form a **model** of the system

If model structure is simple enough, could use mathematical methods to get exact information on questions of interest — *analytical solution*



Systems

Types of systems

- *Discrete*
  - State variables change instantaneously at separated points in time
  - Bank model: State changes occur only when a customer arrives or departs
- *Continuous*
  - State variables change continuously as a function of time
  - Airplane flight: State variables like position, velocity change continuously

Many systems are partly discrete, partly continuous

ADVANTAGES OF SIMULATION

Most complex systems reguire complex models

Uncommon situation

Easy to campare alternatives

Expenditures

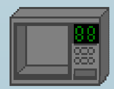
–Build it and see if it works out?

–Simulate current, expanded operations — could also investigate many other issues along the way, quickly and cheaply

Simulation provide better comprehension of inner process

Drawback of Simulation

- **Models of large systems are usually very complex**
  - But now have better modeling software … more general, flexible, but still (relatively) easy to use
- **Stochastic simulations produce only estimates – with noise**
  - Simulations usually produce large volumes of output – need to summarize, statistically analyze
- **Impression that simulation is “just programming”**
  - In appropriate level of detail - tradeoff between model accuracy and computational efficiency
  - Inadequate design and analysis of simulation experiments – simulation need careful design and analysis of simulation models – simulation methodology



Application areas

- Designing and operating **transportation systems** such as airports, freeways, ports, and subways
- Evaluating designs for **service** organizations such as call centers, fast-food restaurants, hospitals, and post offices
- Analyzing financial or **economic systems**

Monte Carlo Simulation

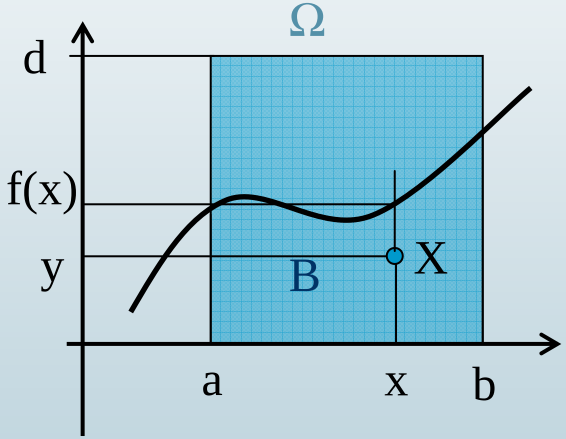
No time element (usually)

- Wide variety of mathematical problems
- Example: Evaluate a “difficult” integral

$$I = \int_a^b f(x)$$

  – Let x~ U(a, b), and let y ~ U(a, b);

  – Algorithm: Generate x ~ U(a, b), let y ~ U(a, b); repeat; integral will be estimate on a relative frequency.



$$\Omega = \langle a, b \rangle \times \langle 0, d \rangle$$

area of Ω: $S_\Omega = d\,|b-a|$

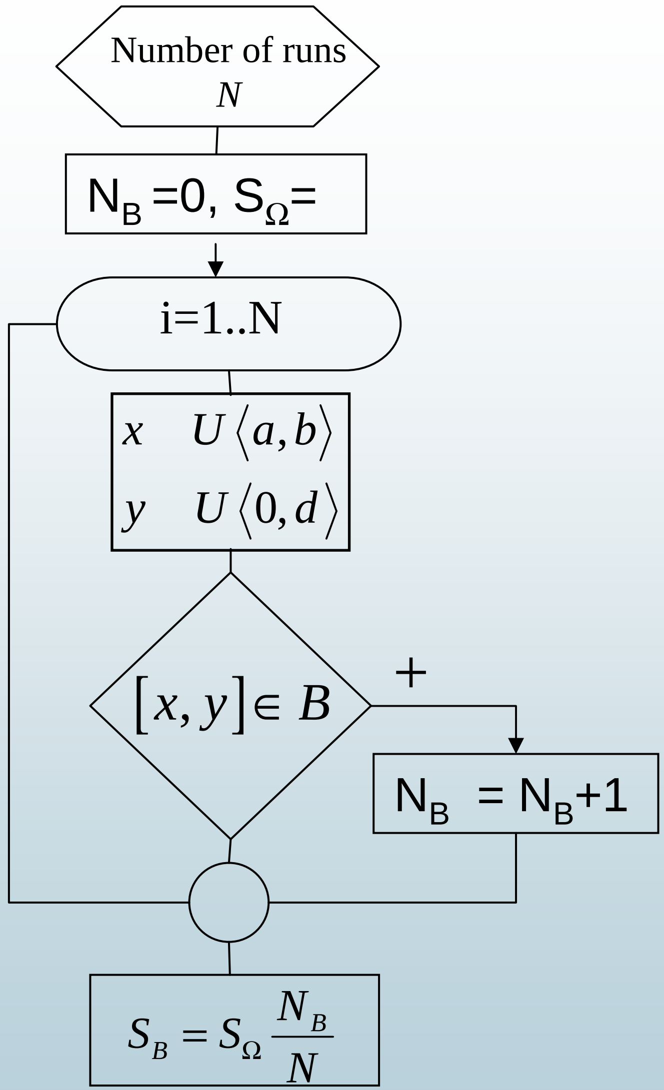
$$I = \int_a^b f(x) = S_B$$

X … random point from Ω $\quad X \in B \Leftrightarrow f(x) > y$

Geometric probability: $P(X \in B) = \dfrac{S_B}{S_\Omega}$

Probability is estimate on frequency $P(X \in B) = \dfrac{N_B}{N_\Omega}$

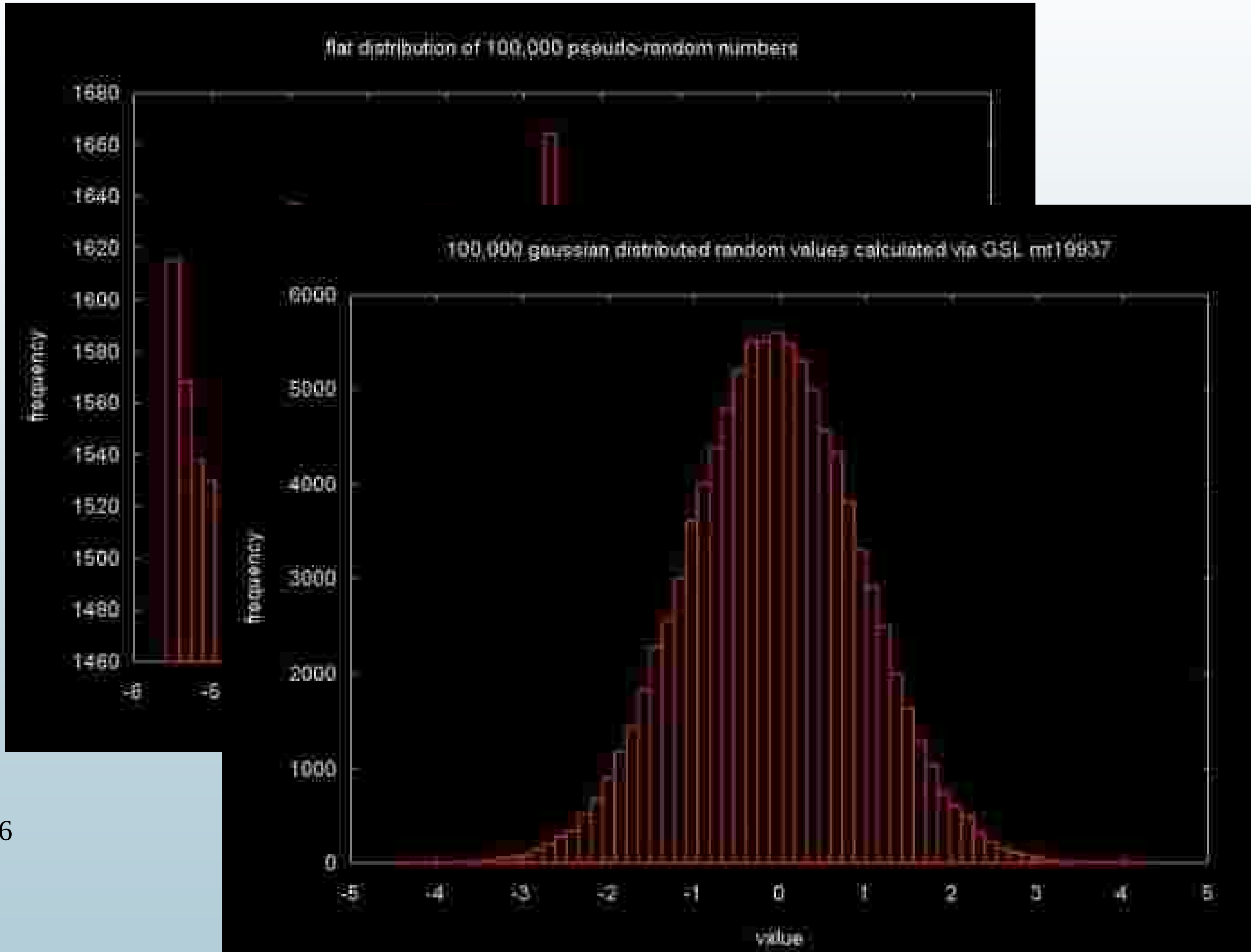
$$S_B = S_\Omega \frac{N_B}{N}$$



0.5069337796517863
-1.1990260943909907
0.26505726444267585
-1.854711766852047
-1.4050150896076032
-0.17275028150144303
-0.3817032401725773
-0.43513693709188666
0.5542356519201573
0.3889663414546052
2.859126301717398
0.9727562577423773
0.5598916815276701
0.38242700896374465
-0.07907552859173163
0.5357541687700696
-0.04447830919338274
1.7795888259824646
-1.4581832070349238
0.12294379539951389
0.657723445793143
-1.5941767391224992
-1.2708787314281355
-0.5238183331104613
-0.36917071169315707
-1.3565030009127183
0.28153491504853023
1.348719082678584
-0.10598961382359845
0.10833847688122822
-1.3677192275044773
-0.051733234063638056
-1.0245136587486698
-0.4719250601619712
2.5932463936402032
-0.5468976666902965

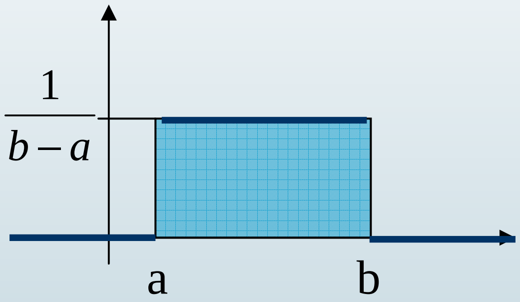
Random variable

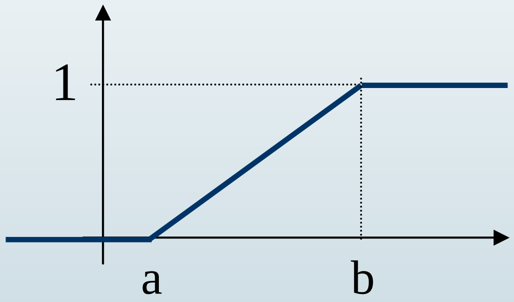
```
Y=rand(n)
```



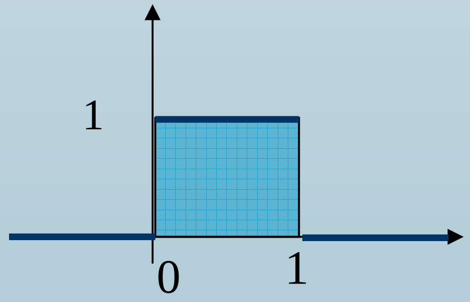
Random number generator

1. Uniform distribution < a ,b >





Suppose, we have random number from unimormly distributed variable $X \quad uniform\langle 0,1\rangle$



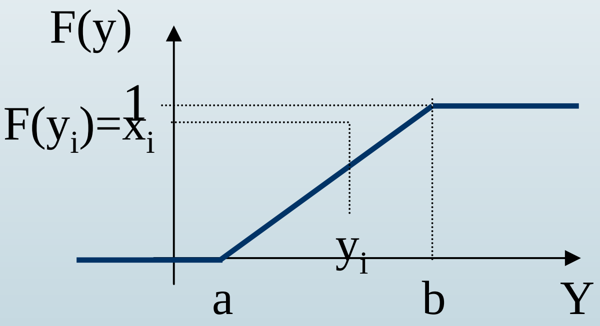
$$Y \quad uniform\langle a,b\rangle$$

$$Y_i = a + (b-a)X_i$$

Universal method for random number transformation

Values of Distribution function $X_i=F(Y_i)$ have a uniform distribution

$$X_i \quad uniform\langle 0,1\rangle$$



$$Y \quad uniform\langle a,b\rangle$$

$$x = F(y) = \frac{y-a}{b-a}$$

$$Y_i = X_i(b-a)+a$$

$$P\left(Y_i < Y_0\right) = F\left(Y_0\right)$$

$$P\left(Y_i < F^{-1}(X_0)\right) = X_0$$

$$P\left(F\left(Y\right)_i < X_0\right) = X_0$$

$$P\left(X_i < X_0\right) = X_0$$

$$F\left(X_0\right) = X_0$$

2. Exponential distribution

$$f(y) = \lambda e^{-\lambda y}$$

$$F(y) = 1 - e^{-\lambda y}$$

$$x = 1 - e^{-\lambda y}$$

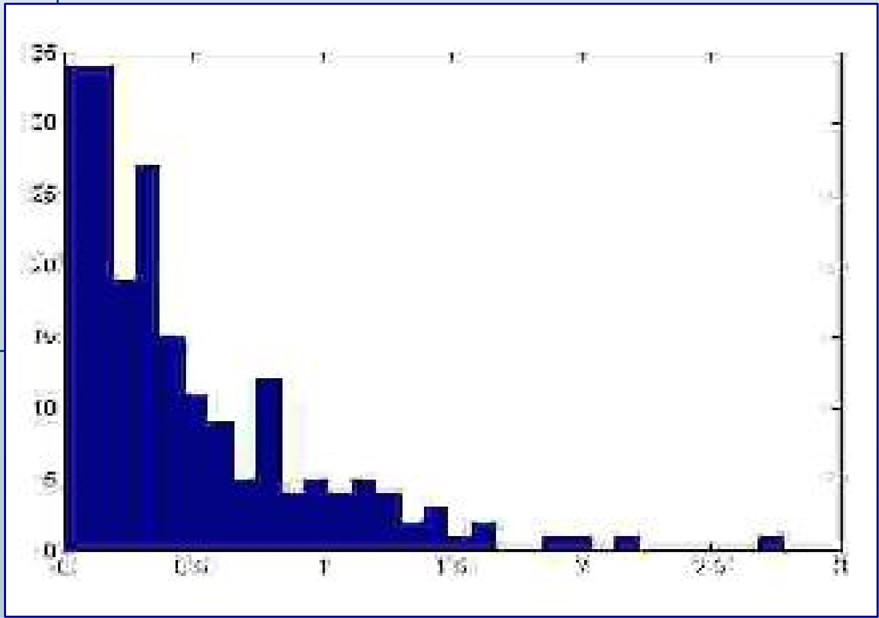
$$e^{-\lambda y} = 1 - x$$

$$-\lambda y = \ln(1 - x)$$

$$y = \frac{\ln(1 - x)}{-\lambda}$$

```
function y=randexp(n,b)
%random variable~exp(b)
for i=1:n
 x(i)=rand;
 y(i)=-log(1-x(i))/b;
End
```

```
>> delay=randexp(200,2);
>> hist(delay,30)
```



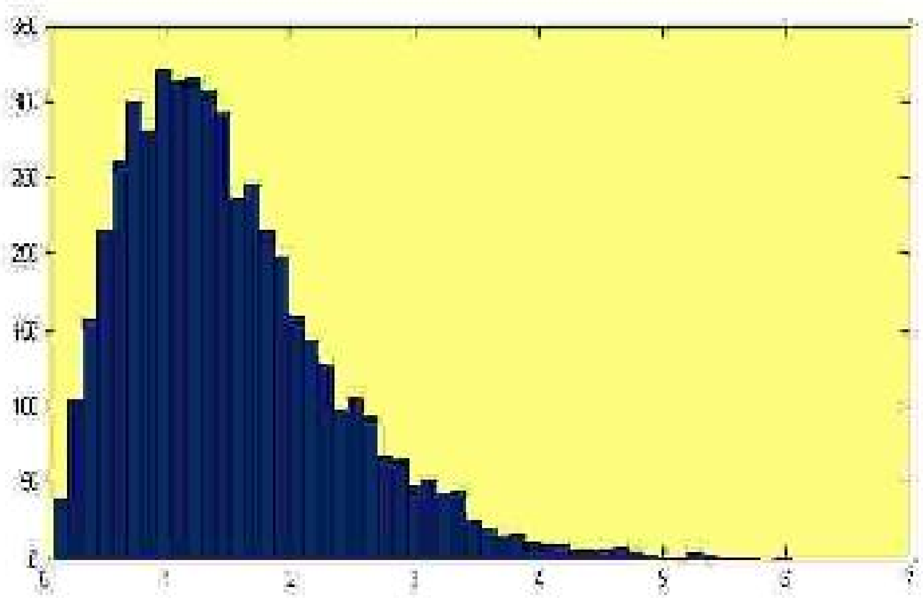
3. Erlang distribution

$$f(y) = \frac{\lambda^k y^{k-1}}{(k-1)!} e^{-\lambda y}$$

součet k nezávislých náhodných veličin s exponenciálním rozdělením

```
function y=erlang(n,b,k)
%vector(n,1) of random
%variable ~ erlang(b,k)
for i=1:n
  y(i)=0
  for j=1:k
    x(j)=randexp(1,b);
    y(i)=y(i)+x(j);
  end
end
```

```
>> delay=erlang(5000,2,3);
>> hist(delay,50)
```



4. Normal normální rozdělení

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

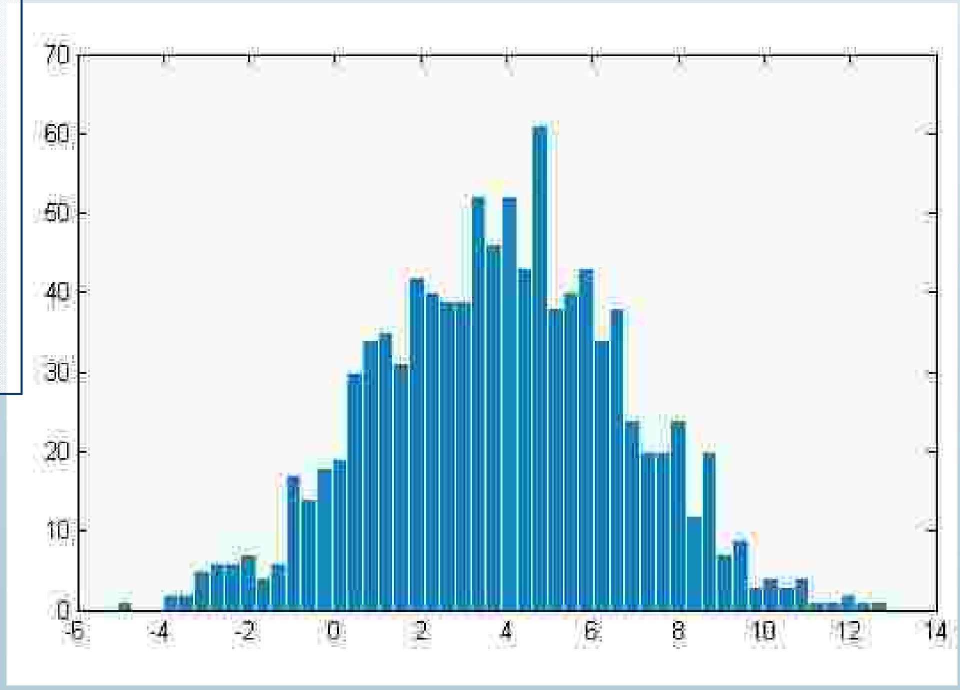
$$X \quad N(0,1) \qquad Y \quad N(\mu,\sigma^2)$$

$$Y = \mu + \sigma X$$

```
X=randn(n)
```

```
>> delay=norm_ab(1000,4,2);
>> hist(delay,30)
```

```
function y=norm_ab(n,s)
%vector(n,1) of normal
%variable ~ N(a,b)
x=randn(n,1);
y=x*s+n
```



4. Raylleigh distribution

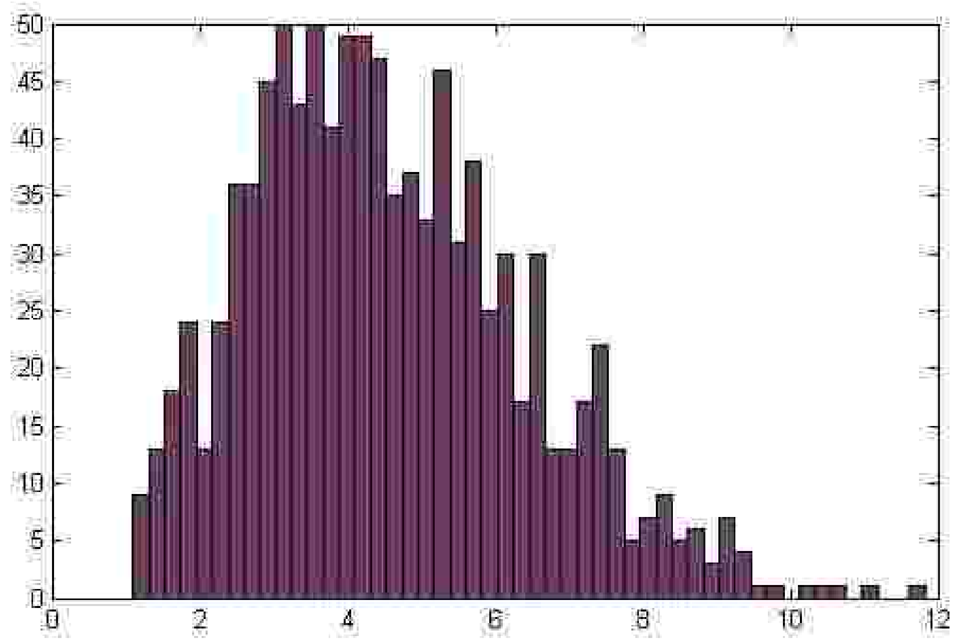
$$F(y) = 1 - e^{-\frac{(y-a)^2}{c^2}}$$

$$Y_i = a + c\sqrt{-\ln X_i}$$

$$f(t) = \frac{2(y-a)}{c^2} \cdot e^{-\frac{(y-a)^2}{c^2}}$$

```
>> delay=norm(1000,1,4);
>> hist(delay,30)
```

```
function y=randreyll(n,a,c)
%vector(n,1) of Reyll's
%variable ~ R(a,c)
x=rand(n,1);
y=c*sqrt(-log(x))+a;
```

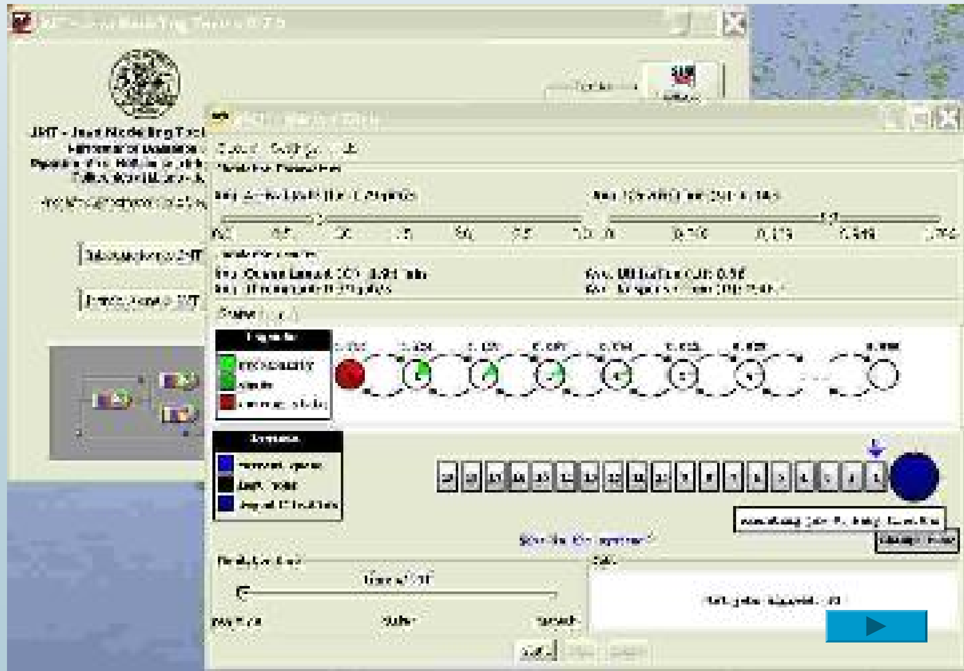


Time-Advance Mechanisms

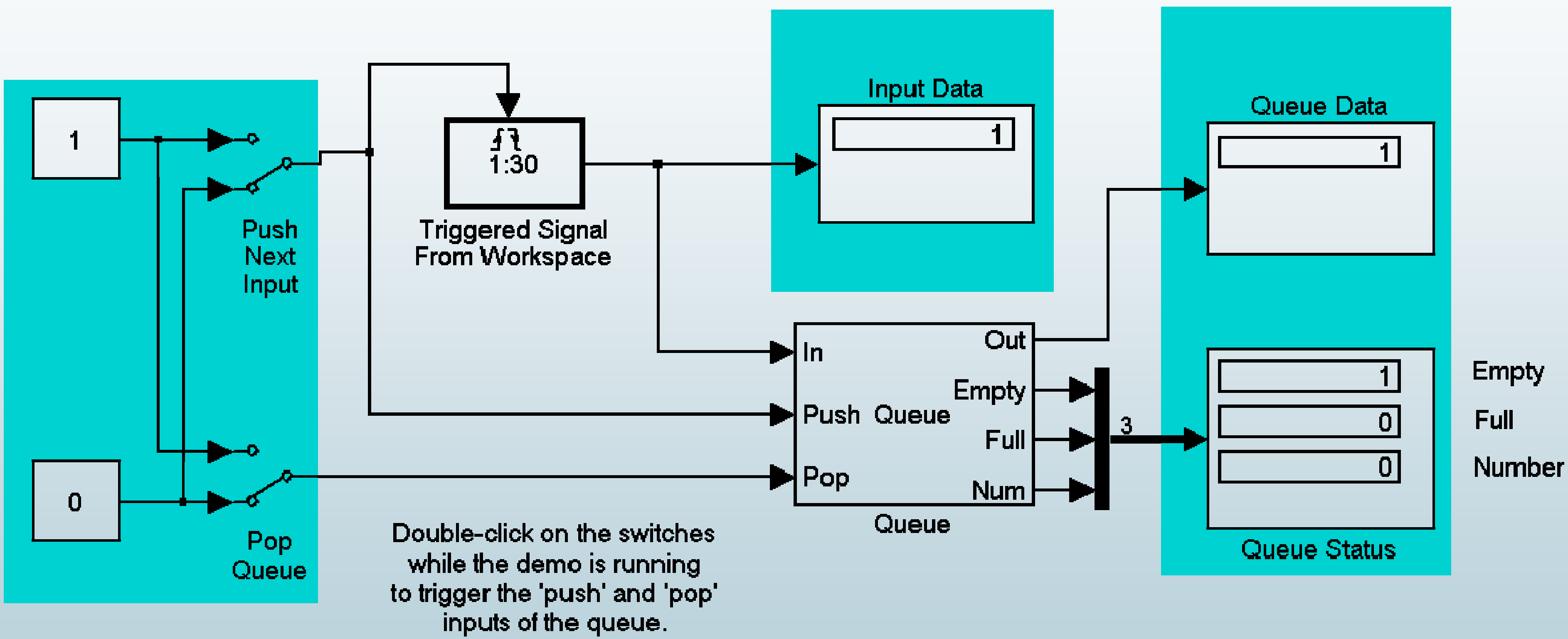
Simulation clock: Variable that keeps the current value of (simulated) time in the model

Two approaches for time advance

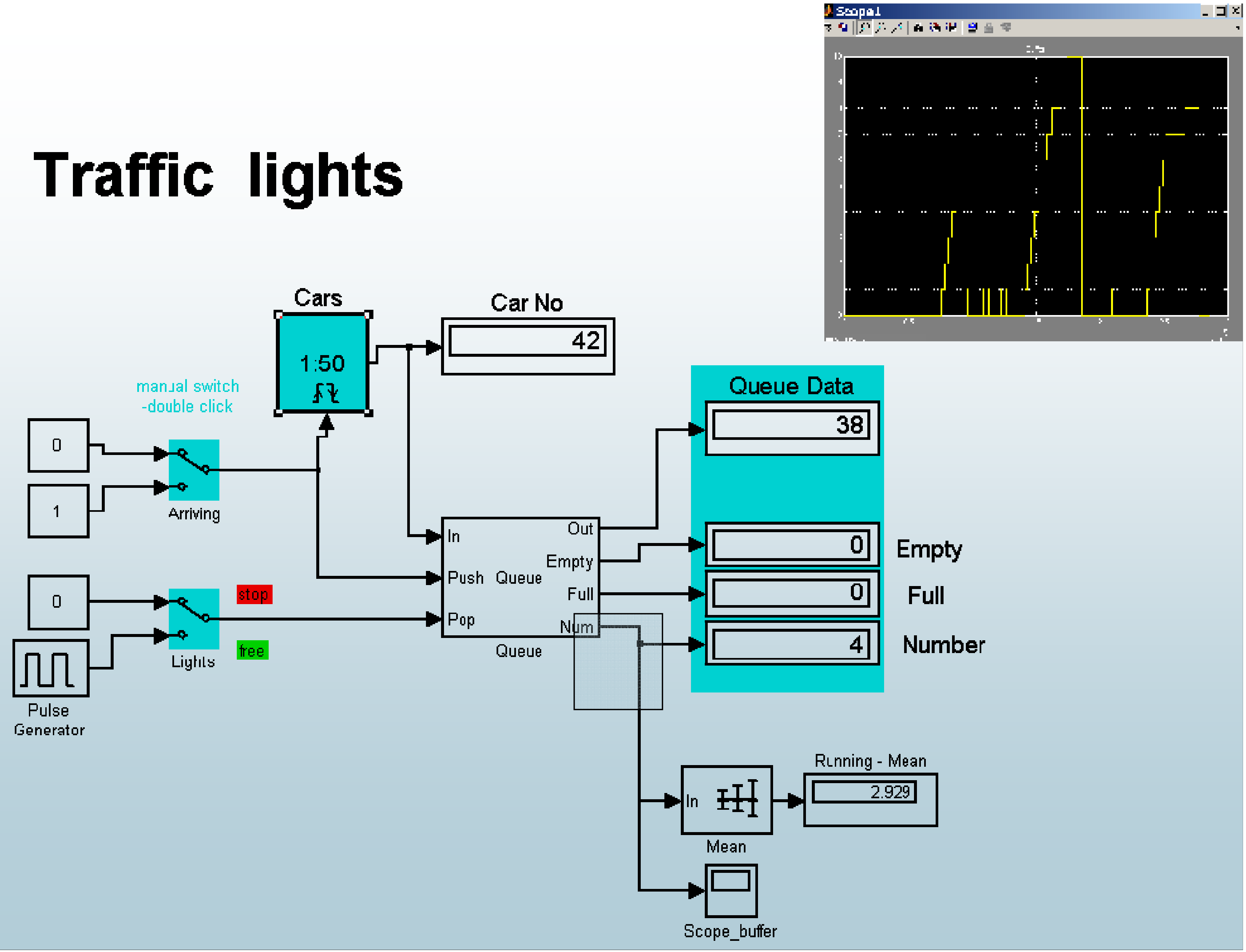
- *Next-event time advance –*
  Event: Instantaneous occurrence that <u>may</u> change the state of the system
- *Fixed-increment time advance*

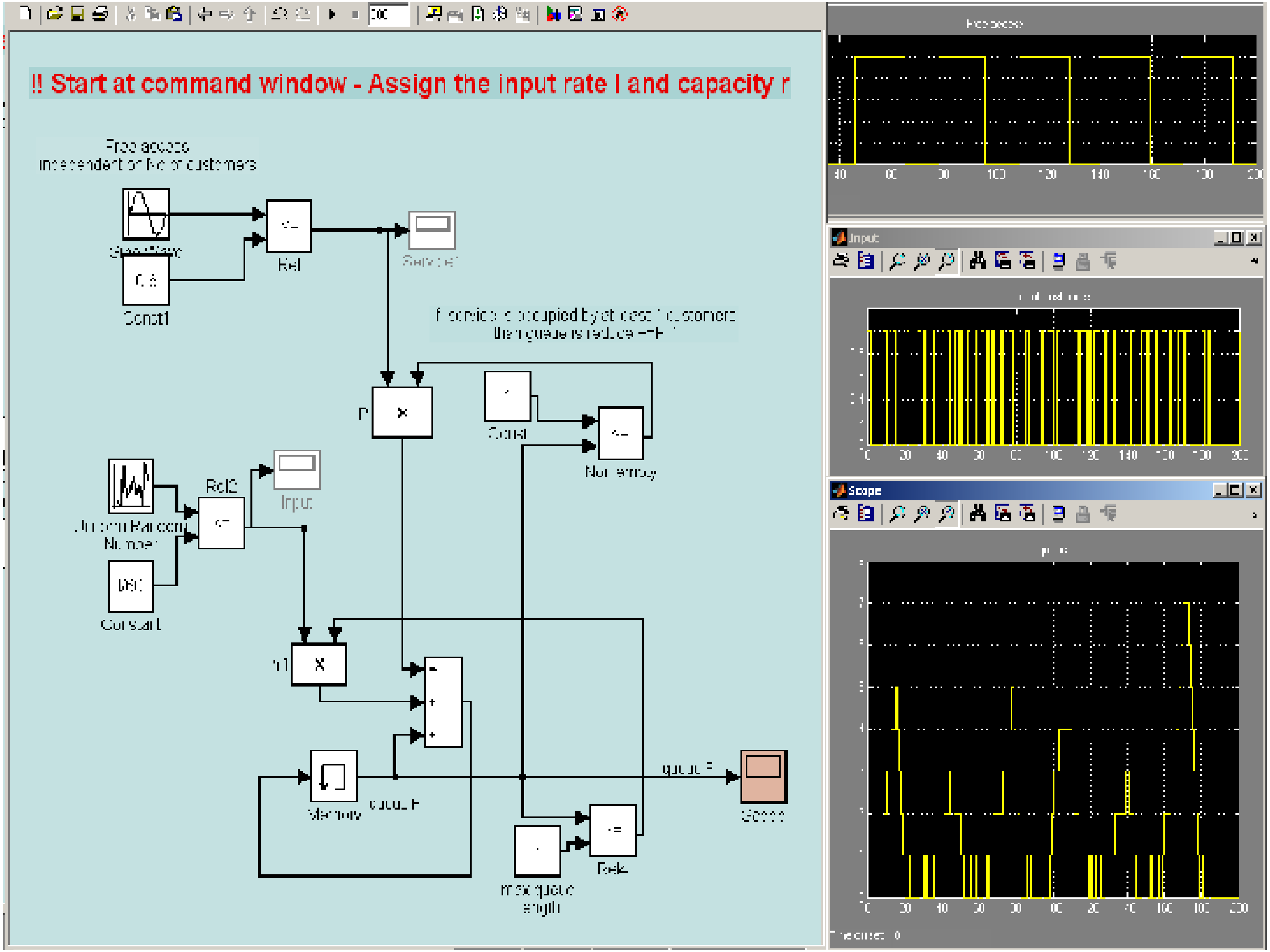


Matlab - Simulink



Traffic lights




!! Start at command window - Assign the input rate l and capacity r
Const1
Rel
Input
Const
Non empty
Scope
Memory
Scope
Input
Uniform Random Number
Constant