BAYESIAN OPTIMUM RADAR DETECTOR IN NON-GAUSSIAN NOISE

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ABSTRACT

In this paper, a theoretical expression of the optimum non-Gaussian radar detector is derived from the non-Gaussian SIRP model (Spherically Invariant Random Process) clutter and a bayesian estimator of the *characteristic function* of the SIRP. The SIRP model is used to perform coherent detection and to modelize the clutter as a complex Gaussian process whose variance is itself a positive random variable (r.v.). The PDF of the variance characterizes the statistics of the SIRP and after performing a bayesian estimation of this PDF from reference clutter cells we derive the Bayesian Optimum Radar Detector (BORD) and its statistical asymptotic form without any knowledge about the statistics of the clutter. We evaluate BORD performance for an unknown target signal embedded in K-distributed clutter and compare with optimum detectors performance (such as Optimum K Detector - OKD - in K-distributed clutter).

1. INTRODUCTION

Coherent radar detection against non-Gaussian clutter has gained many interests in the radar community since experimental clutter measurements made by organizations like MIT [4] have shown to fit non-Gaussian statistical models. One of the most tractable and elegant non-Gaussian model results in the so-called *Spherically Invariant Random Process* (SIRP) theory which states that some non-Gaussian random processes are the product of a Gaussian random process with a non-negative random variable (r.v.) (the variance of the Gaussian process is itself a r.v.). This model is the base of many results like Gini et al.'s works [5] in which is derived the optimum detector in the presence of composite disturbance of known statistics modeled as SIRP.

In this paper, a bayesian approach is proposed to determine the PDF of the variance (the characteristic function of the SIRP) from N reference clutter cells. We use the Bayes'rule and a Monte Carlo integration given a *non informative* prior on the variance PDF. This approach exploits the SIRP model particularity to describe non-Gaussian processes as compound processes and allows to derive the expression of the optimum detector called Bayesian Optimum Radar Detector (BORD). Henceforth, it is no more necessary to have any knowledge about the clutter statistics and the BORD deals directly with the received data. In section 2 and 3, we briefly recall the formulation of a detection problem and describe how the SIRP model clutter allows to derive general and particular optimum detector. In section 4, we explain the bayesian approach used to determine a bayesian estimator to the variance PDF and give the expression of the resulting BORD. Section 5 is devoted to the simulations description to evaluate BORD performance (compared with optimum detectors performance). Conclusions and outlooks are given in section 6.

2. GENERAL RELATIONS OF DETECTION THEORY

We consider here the basic problem of detecting the presence (H_1) or absence (H_0) of a complex signal $\mathbf{s}$ in a set of N measurements of m-complex vectors $\mathbf{y} = \mathbf{y}_\mathrm{I} + \mathrm{j}\,\mathbf{y}_\mathrm{Q}$ corrupted by a sum $\mathbf{c}$ of independent additive complex noises (noises + clutter). The problem can be described in terms of a statistical hypothesis test :

$$H_0 : \mathbf{y} \;=\; \mathbf{c} \tag{1}$$
$$H_1 : \mathbf{y} \;=\; \mathbf{s} + \mathbf{c} \tag{2}$$

When present, the target signal $\mathbf{s}$ corresponds to a modified version of the perfectly known emitted signal $\mathbf{t}$ and can be rewritten as $\mathbf{s} = A\,T(\underline{\theta})\,\mathbf{t}$. A is the target amplitude. We suppose determined all the others parameters ($\underline{\theta}$) which characterize the target (Doppler frequency, time delay, ...). In the following, we will note $\mathbf{p} = T(\underline{\theta})\,\mathbf{t}$. The observed vector $\mathbf{y}$ is used to form the Likelihood Ratio Test (LRT) $\Lambda(\mathbf{y})$ which is compared with a threshold η set to a desired false alarm probability (P_{fa}) value :

$$\Lambda(\mathbf{y}) = \frac{p_\mathbf{y}(\mathbf{y}/H_1)}{p_\mathbf{y}(\mathbf{y}/H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta \tag{3}$$

The LRT performances follow from the statistics of the data. P_{fa} is the probability of choosing H_1 when the target is absent, and the detection probability (P_d) is the probability of choosing H_1 when the target is present, that is :

$$P_{fa} = \mathbb{P}(\Lambda(\mathbf{y}) \underset{H_0}{>} \eta) \quad \text{and} \quad P_d = \mathbb{P}(\Lambda(\mathbf{y}) \overset{H_1}{>} \eta) \tag{4}$$

3. NON-GAUSSIAN CLUTTER CASE : SIRV AND OPTIMUM RADAR DETECTOR

In the case of non-Gaussian clutter, detection strategies can be derived if we consider a particular clutter nature, i.e. if an *a priori* hypothesis is made on the clutter statistic. To model non-Gaussian clutter and derive general detector expressions, we use the SIRP representation [3, 9, 10].

3.1. Description and general expressions

The SIRV model interprets each element of the clutter vector $\mathbf{c}$ as the product of a m-complex Gaussian vector $\mathbf{x}$ ($\mathcal{CN}(\mathbf{0}, 2\,\mathbf{M})$) with a positive r.v. τ, that is $\mathbf{c} = \mathbf{x}\,\sqrt{\tau}$.

The PDF of the variable τ is the so-called *characteristic function* of the SIRV and the so formed vector $\mathbf{c}$ is, conditionally to τ, a complex Gaussian random vector ($\mathcal{CN}(\mathbf{0}, 2\,\tau\,\mathbf{M})$) with joint PDF $p(\mathbf{c}/\tau)$. The marginal PDF of the clutter is then :

$$p(\mathbf{c}) = \int_0^{+\infty} \frac{1}{(2\pi\tau)^m |\mathbf{M}|} \exp\left(-\frac{\mathbf{c}^\dagger \mathbf{M}^{-1}\mathbf{c}}{2\tau}\right) p(\tau) d\tau. \tag{5}$$

where $\dagger$ is the transpose conjugate operator and $|\mathbf{M}|$ is the determinant of the matrix $\mathbf{M}$. This general expression allows to determine, for a known $p(\tau)$, the joint PDFs of non-Gaussian random vectors. For example, the joint K-distributed PDF is obtained if $p(\tau)$ is a Gamma PDF (see further).

3.2. SIRP Optimum Detector

Applied to the detection problem,the expression (5) gives $p_{\mathbf{c}}(\mathbf{y}/H_0)$ and $p_{\mathbf{c}}(\mathbf{y}/H_1) = p_{\mathbf{c}}(\mathbf{y} - \mathbf{s}/H_0)$ when the target signal $\mathbf{s}$ is known. The LRT becomes (with the same notations as in [5]) :

$$\int_0^{+\infty} \left[\exp\left(-\frac{q_1(\mathbf{y})}{2\tau}\right) - \exp\left(\lambda - \frac{q_0(\mathbf{y})}{2\tau}\right)\right] \frac{p(\tau)}{\tau^m}\, d\tau \underset{H_0}{\overset{H_1}{\gtrless}} 0 \tag{6}$$

where $q_0(\mathbf{y}) = \mathbf{y}^\dagger \mathbf{M}^{-1}\mathbf{y}$, $q_1(\mathbf{y}) = q_0(\mathbf{y} - \mathbf{s})$ for a known signal $\mathbf{s}$ and $\lambda = \ln(\eta)$.

When the target signal $\mathbf{s}$ is unknown, ML estimation of A is performed and the detection strategy is given by (6) where now :

$$q_1(\mathbf{y}) = \mathbf{y}^\dagger \mathbf{M}^{-1}\mathbf{y} - \frac{|\mathbf{p}^\dagger \mathbf{M}^{-1}\mathbf{y}|^2}{\mathbf{p}^\dagger \mathbf{M}^{-1}\mathbf{p}}. \tag{7}$$

With (7), the (6) expression is called Generalized LRT (GLRT).

3.3. Example : Optimum K Detector : the OKD

In the case of K-distributed clutter (size m) with parameters ν and b, the random variable τ is Gamma(ν,$\beta = 2/b^2$)-distributed with PDF expression :

$$p(\tau) = \frac{\tau^{\nu-1}}{\Gamma(\nu)\,\beta^\nu} \exp\left(-\frac{\tau}{\beta}\right). \tag{8}$$

and the PDF of $\mathbf{y}$ under H_0 hypothesis is given by :

$$p_{\mathbf{y}}(\mathbf{y}/H_0) = \frac{2\,b^{\nu+m}}{\pi^m\,|\mathbf{M}|\Gamma(\nu)\,2^{\nu+m}}\, q(\mathbf{y})^{\frac{\nu-m}{2}}\, K_{\nu-m}(b\,\sqrt{q(\mathbf{y})}). \tag{9}$$

where $K_\nu(.)$ is the modified Bessel function of order ν, $\Gamma(.)$ is the Gamma function and $q(\mathbf{y}) = \mathbf{y}^\dagger\,\mathbf{M}^{-1}\mathbf{y}$. The value of ν determines the spikiness of the distribution. Following the same processes with (6), the expression of the so-called Optimum K-distributed Detector (OKD) becomes $\forall m \geq 2$:

$$\left(\frac{q_1(\mathbf{y})}{q_0(\mathbf{y})}\right)^{\frac{\nu-m}{2}} \cdot \frac{K_{\nu-m}\left(b\,\sqrt{q_1(\mathbf{y})}\right)}{K_{\nu-m}\left(b\,\sqrt{q_0(\mathbf{y})}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta, \tag{10}$$

where $q_0(\mathbf{y}) = \mathbf{y}^\dagger\,\mathbf{M}^{-1}\mathbf{y}$ and $q_1(\mathbf{y})$ is given by (7) for unknown signal $\mathbf{s}$. For $m = 1$, the expression is given by [7] :

$$(q_0(\mathbf{y}))^{\frac{\nu-1}{2}}\, K_{\nu-1}\left(b\,\sqrt{q_0(\mathbf{y})}\right) \underset{H_0}{\overset{H_1}{\lessgtr}} \frac{(2\,\beta)^{\frac{\nu-1}{2}}\,\Gamma(\nu)}{2\,\nu\,\eta} \tag{11}$$

4. BAYESIAN OPTIMUM RADAR DETECTOR

In this section, we propose to use a bayesian estimator for the characteristic function of the SIRV which comes from the Bayes'rule and Monte Carlo integration. Then, the BORD expression is derived.

4.1. Bayesian Study of the Problem

As it was said in the previous section, for a known variance PDF $p(\tau)$, it is possible to derive the associated detector expression. The idea of a bayesian approach is to determine, from N clutter reference cells of size m, $\mathbf{R} = [\mathbf{r}_1, \ldots, \mathbf{r}_N]^T$ where $\mathbf{r}_i = [r_i(1), \ldots, r_i(m)]^T$, a bayesian estimator $\hat{p}(\tau)$ of the variance PDF $p(\tau)$. We write $p(\tau)$ as follows :

$$p(\tau) = \int_{\mathbb{R}^m} p(\tau/\mathbf{r})p(\mathbf{r})\, d\mathbf{r}, \tag{12}$$

Given $\mathbf{r}_{i=1}^N$ a Monte Carlo estimation of (12) is :

$$\hat{p}(\tau) = \frac{1}{N}\sum_{i=1}^{N} p(\tau/\mathbf{r}_i). \tag{13}$$

The Bayes'rule provides us :

$$p(\tau/\mathbf{r}_i) = \frac{p(\mathbf{r}_i/\tau)\,g(\tau)}{p(\mathbf{r}_i)}, \tag{14}$$

where $g(\tau)$ is the prior distribution of τ for the reference cells and the equation (13) becomes :

$$\hat{p}(\tau) = \frac{1}{N}\sum_{i=1}^{N} \frac{p(\mathbf{r}_i/\tau)\,g(\tau)}{p(\mathbf{r}_i)}. \tag{15}$$

The normalization constant $p(\mathbf{r}_i)$ is obtained by integrating the numerator in (13) over $g(\tau)$ and is given by :

$$p(\mathbf{r}_i) = \int_0^{+\infty} p(\mathbf{r}_i/\tau)\,g(\tau) d\tau. \tag{16}$$

Replacing $\hat{p}(\tau)$ in (6), the expression of the Bayesian Optimum Radar Detector can be derived. It is no more necessary to have knowledge about the clutter statistics or to estimate the variance PDF thanks to a Padé approximation as we did in [7] for the PEOD.

4.2. BORD Expression

The N reference clutter cells $[\mathbf{r}_1, \ldots, \mathbf{r}_N]^T$ are supposed to be modeled as SIRV and so we have :

$$p(\mathbf{r}_i/\tau) = \frac{1}{(2\pi)^m\,|\mathbf{M}|}\,\tau^{-m}\,\exp\left(-\frac{\mathbf{r}_i^\dagger\mathbf{M}^{-1}\mathbf{r}_i}{2\tau}\right), \quad (17)$$

As the clutter statistics is unknown, we use a non-informative prior distribution $g(\tau) = 1/\tau$ to retrieve the *a posteriori* PDF of τ given the N reference cells and (14) becomes :

$$p(\tau/\mathbf{r}_i) = \frac{1}{(2\pi)^m\,|\mathbf{M}|\,p(\mathbf{r}_i)}\,\tau^{-m-1}\,\exp\left(-\frac{\mathbf{r}_i^\dagger\mathbf{M}^{-1}\mathbf{r}_i}{2\tau}\right). \quad (18)$$

The normalization constant $p(\mathbf{r}_i)$ is computed as follows :

$$\begin{aligned} p(\mathbf{r}_i) &= \int_0^{+\infty} p(\mathbf{r}_i/\tau)\,g(\tau)d\tau \\ &= \int_0^{+\infty} \frac{1}{(2\pi)^m\,|\mathbf{M}|}\,\tau^{-m-1}\,\exp\left(-\frac{\mathbf{r}_i^\dagger\mathbf{M}^{-1}\mathbf{r}_i}{2\tau}\right)d\tau \\ &= \frac{\Gamma(m)}{\pi^m\,|\mathbf{M}|\,(\mathbf{r}_i^\dagger\mathbf{M}^{-1}\mathbf{r}_i)^m}. \end{aligned} \quad (19)$$

The equation (18) becomes :

$$p(\tau/\mathbf{r}_i) = \frac{(\mathbf{r}_i^\dagger\mathbf{M}^{-1}\mathbf{r}_i)^m}{2^m\,\Gamma(m)}\,\tau^{-m-1}\,\exp\left(-\frac{\mathbf{r}_i^\dagger\mathbf{M}^{-1}\mathbf{r}_i}{2\tau}\right), \quad (20)$$

which is exactly an Inverse Gamma PDF $h_i(\tau)$ with parameters m and $2/\mathbf{r}_i^\dagger\mathbf{M}^{-1}\mathbf{r}_i$. So, we have :

$$\hat{p}(\tau) = \frac{1}{N}\sum_{i=1}^{N} h_i(\tau). \quad (21)$$

The BORD expression which is given for each observation cell $\mathbf{y}_{obs}$ (size m) and given the N reference clutter vectors $\mathbf{r}_{i=1}^N$ becomes after the integration of (6) over $\hat{p}(\tau)$:

$$\Lambda(\mathbf{y}_{obs}) = \frac{\displaystyle\sum_{i=1}^{N}\left[\frac{\mathbf{r}_i^\dagger\,\mathbf{M}^{-1}\,\mathbf{r}_i}{(q_1(\mathbf{y}_{obs}) + \mathbf{r}_i^\dagger\,\mathbf{M}^{-1}\,\mathbf{r}_i)^2}\right]^m}{\displaystyle\sum_{i=1}^{N}\left[\frac{\mathbf{r}_i^\dagger\,\mathbf{M}^{-1}\,\mathbf{r}_i}{(q_0(\mathbf{y}_{obs}) + \mathbf{r}_i^\dagger\,\mathbf{M}^{-1}\,\mathbf{r}_i)^2}\right]^m} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda, \quad (22)$$

where the two quadratic forms q_0 and q_1 are the same that for OKD (cf. 7).

The BORD expression depends only on the reference clutter cells which provide all the necessary information about the clutter statistics making itself "self-adaptive". Moreover, the BORD structure contains the classical Gaussian matched filter (in the quadratic form $q_1(\mathbf{y}_{obs})$) that means that it can be implemented in conventional radar systems to improve their detection performance.

4.3. Asymptotical Result of the BORD

The BORD expression comes after a Monte Carlo estimation of (12) given N reference clutter vector $\mathbf{r}_{i=1}^N$. Given $Z_i = \mathbf{r}_i^\dagger\mathbf{M}^{-1}\mathbf{r}_i$ which is corresponding to a positive r.v. with PDF $p(Z)$, the BORD expression can be considered as the Monte Carlo estimation of :

$$\frac{\displaystyle\int_0^{+\infty}\frac{Z^m}{(q_1(\mathbf{y}_{obs}) + Z)^{2\,m}}\,p(Z)\,dZ}{\displaystyle\int_0^{+\infty}\frac{Z^m}{(q_0(\mathbf{y}_{obs}) + Z)^{2\,m}}\,p(Z)\,dZ}. \quad (23)$$

Given that $\mathbf{r} = \sqrt{\tau}\,\mathbf{x}$ where $\mathbf{x}$ is a complex Gaussian vector of size m with a covariance matrix $2\,\mathbf{M}$, we have $Z = \mathbf{r}^\dagger\mathbf{M}^{-1}\mathbf{r} = \tau\,\mathbf{x}^\dagger\mathbf{M}^{-1}\mathbf{x}$. The quadratic form $Q = \mathbf{x}^\dagger\mathbf{M}^{-1}\mathbf{x}$ is χ^2_{2m} distributed ($\chi^2_{2m} = \mathcal{G}(m, 2)$). So, Z/τ is $\mathcal{G}(m, 2\tau)$ and the PDF of Z is derived by integrating $p(Z/\tau)$ over the prior $g(\tau)$. We obtain :

$$\lim_{N\to+\infty}\Lambda_N(\mathbf{y}_{obs}) = \left(\frac{q_0(\mathbf{y}_{obs})}{q_1(\mathbf{y}_{obs})}\right)^m \quad (24)$$

This asymptotical result [8] coincides with the GLRT given, for example, in ([11]), which was obtained by replacing, in the optimum detection structure (6), the two Maximum Likelihood estimates of τ (the one under H_0 and the other one under H_1), where τ is considered as an unknown parameter.

5. SIMULATIONS

We consider here uncorrelated clutter, i.e. the correlation matrix $\mathbf{M}$ is diagonal and known. We compare the BORD performances with those of optimum detectors such OKD and OGD (Optimum Gaussian Detector, optimum for Gaussian clutter) for an unknown target signal embedded in K-distributed clutter. In this case, the OKD is optimum and we see that the BORD performances reach the OKD performances whatever the value of ν is. Different values of the shape parameter are tested, $\nu = 0.1, 0.5, 2, 20$. When $\nu \longrightarrow +\infty$ K-PDF tends to a Gaussian PDF which is confirmed on the series of figures (1),(2), (3) and (4). All the curves represent the detection probability P_d versus the Signal-to-Noise-Ratio (SNR) given for one pulse. As $m = 10$ pulses are considered, the total SNR is 10 $\log_{10}(m) = 10$ dB more than for one pulse.

6. CONCLUSIONS

The present paper has addressed a bayesian approach to the determination of the clutter statistics when the clutter vector is modeled as a SIRV. By this way, a bayesian estimator of the *characteristic function* of the SIRV has been derived from reference clutter cells and the resulting BORD expression depends only on these reference cells. For example, in the case of CFAR (Constant False Alarm Rate) detector, the reference clutter cells are the cells adjacent to the cell under test. In this paper, the detection threshold is derived by Monte Carlo method to reach a desired P_{fa} value. In further work, we will proceed to a Padé approximation to estimate the BORD PDF, that is to say the detection test PDF, and we will study the influence of the number of available cells on detection threshold value.

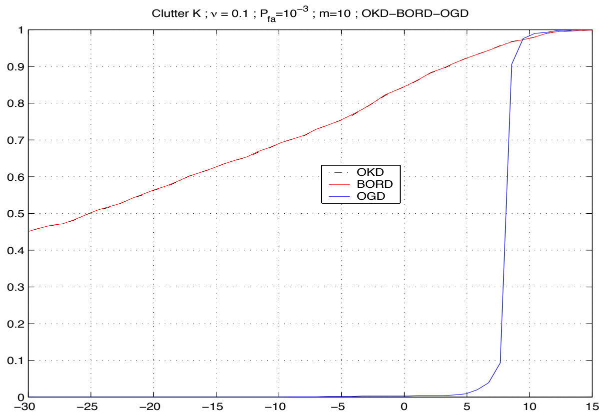

Fig. 1. Performances comparison between the OGD, OKD and BORD for K-distributed clutter ($\nu = 0.1$, $P_{fa} = 10^{-3}$, $m = 10$). OKD and BORD curves are perfectly identical.

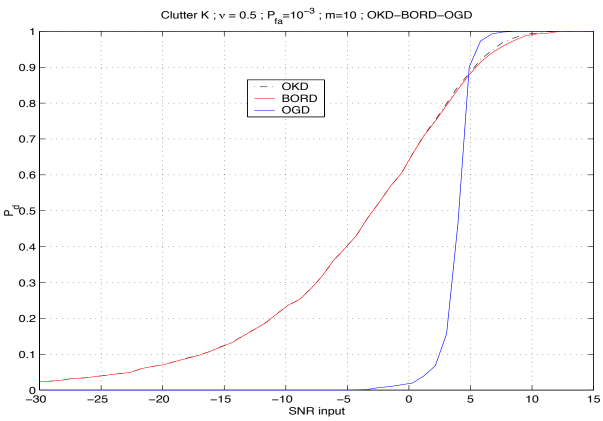

Fig. 2. Performances comparison between the OGD, OKD and BORD for K-distributed clutter ($\nu = 0.5$, $P_{fa} = 10^{-3}$, $m = 10$). OKD and BORD curves are perfectly identical

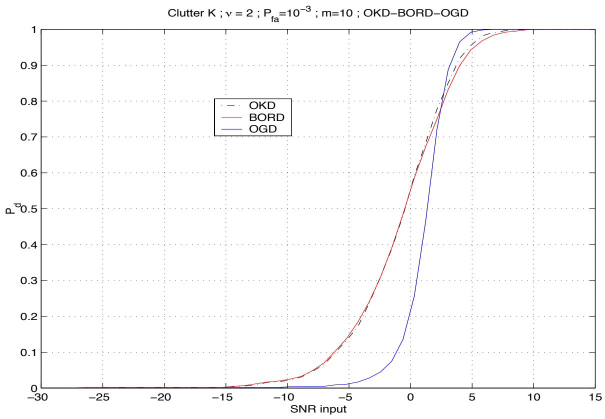

Fig. 3. Performances comparison between the OGD, OKD and BORD for K-distributed clutter ($\nu = 2$, $P_{fa} = 10^{-3}$, $m = 10$). OKD and BORD curves are nearly identical

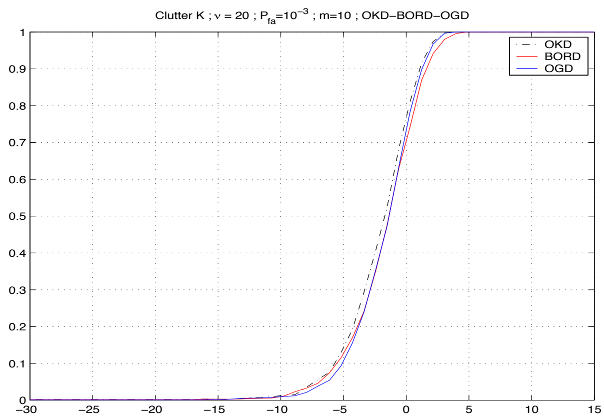

Fig. 4. Performances comparison between the OGD, OKD and BORD for K-distributed clutter ($\nu = 20$, $P_{fa} = 10^{-3}$, $m = 10$). OKD and BORD curves are nearly identical

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