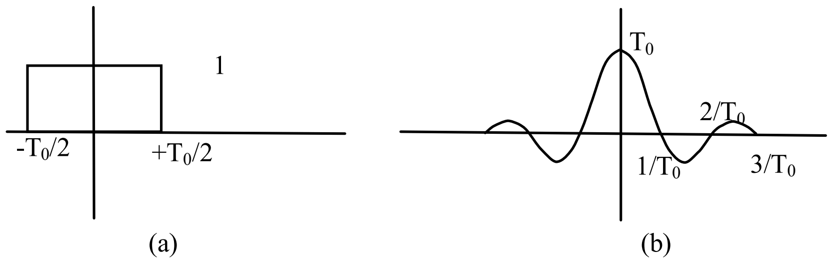

Fig. 1.5 (**a**) A rectangular function of time with the $t = 0$ axis so chosen that the function is an even function. The duration of the signal is $2T_0$. (**b**) Fourier transform of the function.

The Fourier transform of this time function is given by

$$X(f) = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft}\mathrm{d}t = \int_{t_1}^{t_1+T_0} e^{-j2\pi ft}\mathrm{d}t = e^{j2\pi f(t_1+\frac{T_0}{2})}T_0\frac{\sin(2\pi f\frac{T_0}{2})}{(2\pi f\frac{T_0}{2})}.$$

The first term in the Fourier transform is a phase shift factor and has been omitted from the plot in Figure 1.5b for convenience. If the rectangular wave is centered at the origin, $t_1 = -T_0/2$, and the phase shift factor vanishes. This is also in keeping with Property 2 of the Fourier transform given above, which states that the Fourier transform of a real even function must be real and even function of frequency.

1.4 Sampled data and aliasing

Sampled data from input signals are the starting point of digital signal processing. The computation of phasors of voltages and currents begins with samples of the waveform taken at uniform intervals $k\Delta T$, ($k = 0, \pm 1, \pm 2, \pm 3, \pm 4, \cdots\cdots$ }. Consider an input signal $x(t)$ which is being sampled, yielding sampled data $x(k\Delta T)$. We may view the sampled data as a time function $x'(t)$ consisting of uniformly spaced impulses, each with a magnitude $x(k\Delta T)$:

$$x'(t) = \sum_{k=-\infty}^{\infty} x(k\Delta t)\,\delta(t - k\Delta T) \tag{1.11}$$

It is interesting to determine the Fourier transform of the sampled data function given by Eq. (1.11). Note that the sampled data function is a product of the function $x(t)$ and the sampling function $\delta(t - k\Delta T)$, the product being interpreted in the sense of Eq. (1.9). Hence the Fourier transform $X'(f)$ of $x'(t)$ is the convolution of the Fourier transforms of $x(t)$ and of the unit impulse train. By Property 6 of Section 1.3, the Fourier transform of the impulse train is

$$\Delta(f)=\frac{1}{\Delta T}\sum_{k=-\infty}^{\infty}\delta(f-\frac{k}{\Delta T}) \quad . \tag{1.12}$$

Hence the Fourier transform of the sampled data function is the convolution of $\Delta(f)$ and $X(f)$

$$\begin{aligned} X'(f)&=\frac{1}{\Delta T}\int_{-\infty}^{+\infty}X(\phi)\sum_{k=-\infty}^{\infty}\delta(f-\frac{k}{\Delta T}-\phi)\,\mathrm{d}\phi \\ &=\frac{1}{\Delta T}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{+\infty}X(\phi)\delta(f-\frac{k}{\Delta T}-\phi)\,\mathrm{d}\phi \\ &=\frac{1}{\Delta T}\sum_{k=-\infty}^{\infty}X(f-\frac{k}{\Delta T}) \end{aligned} \tag{1.13}$$

Once again the order of summation and integration has been reversed (it being assumed that this is permissible), and the integral is evaluated by the use of the sampling property of the impulse function.

The relationship between the Fourier transforms of $x(t)$ and $x'(t)$ are as shown in Figure 1.6. The Fourier transform of $x(t)$ is shown to be band-limited, meaning that it has no components beyond a cut-off frequency f_c. The sampled data has a Fourier transform which consists of an infinite train of the Fourier transforms of $x(t)$ centered at frequency intervals of $(k/\Delta T)$ for all k. Recall that the sampling interval is ΔT, so that the sampling frequency $f_s = (1/\Delta T)$.

If the cut-off frequency f_c is greater than one-half of the sampling frequency f_s, the Fourier transform of the sampled data will be as shown in Figure 1.7. In this case, the spectrum of the sampled data is different from that of the input signal in the region where the neighboring spectra overlap as shown by the shaded region in Figure 1.7. This implies that frequency components estimated from the sampled data in this region will be in error, due to a phenomenon known as "aliasing".

It is clear from the above discussion that in order to avoid errors due to aliasing, the bandwidth of the input signal must be less than half the sampling frequency utilized in obtaining the sampled data. This requirement is known as the "Nyquist criterion".

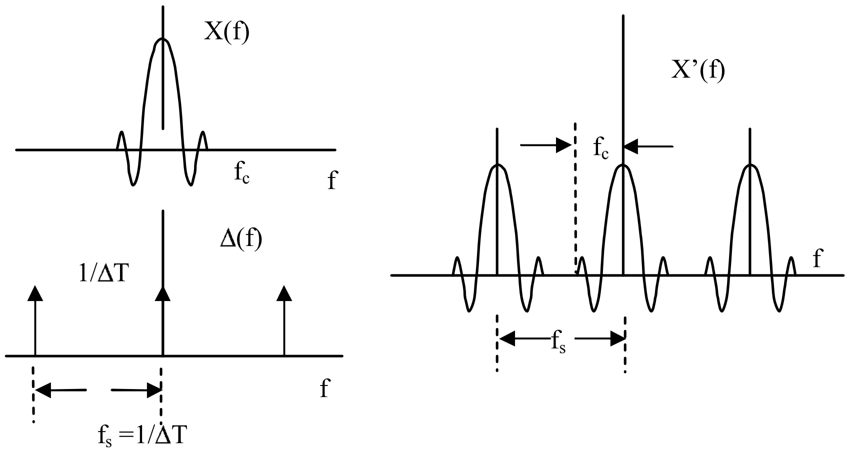

Fig. 1.6 Fourier transform of the sampled data function as a convolution of the Transforms $X(f)$ and $\Delta(f)$. The sampling frequency is f_s, and $X(f)$ is band-limited between $\pm f_c$.

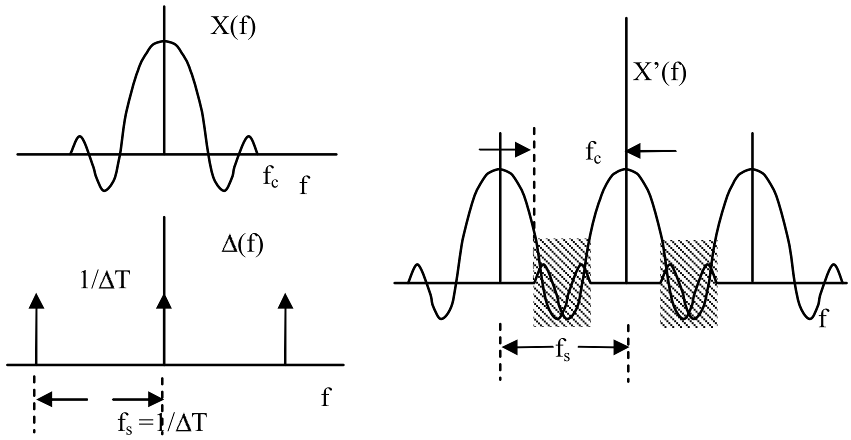

Fig. 1.7 Fourier transform of the sampled data function when the input signal is band-limited to a frequency greater than half the sampling frequency. The estimate of frequencies from sampled data in the shaded region will be in error because of aliasing.

In order to avoid aliasing errors, it is customary in all sampled data systems used in phasor estimation to use anti-aliasing filters which band-limit the input signals to below half the sampling frequency chosen. Note that the signal input cut-off frequency must be *less than* one half the sampling frequency. In practice, the signal is usually band-limited to a value much smaller than the one required for meeting the Nyquist criterion. Anti-aliasing filters are generally passive low-pass R-C filters [11], although active filters may also be used for obtaining a sharp cut-off characteristic. In addition to passive anti-aliasing filters, digital filters may also be used in special cases (e.g., with oversampling and decimation). All anti-aliasing filters introduce frequency-dependent phase shift in the input signal which must be compensated for in determining the phasor representation of the input signal. This will be discussed further in Chapter 5 where the 'Synchrophasor' standard is described.

1.5 Discrete Fourier transform (DFT)

DFT is a method of calculating the Fourier transform of a small number of samples taken from an input signal $x(t)$. The Fourier transform is calculated at discrete steps in the frequency domain, just as the input signal is sampled at discrete instants in the time domain. Consider the process of selecting N samples: $x(k\Delta T)$ with $\{k = 0, 1, 2, \cdots, N-1\}$, ΔT being the sampling interval. This is equivalent to multiplying the sampled data train by a "windowing function" $w(t)$, which is a rectangular function of time with unit magnitude and a span of $N\Delta T$. With the choice of samples ranging from 0 to $N-1$, it is clear that the windowing function can be viewed as starting at $-\Delta T/2$ and ending at $(N-1/2)\Delta T$. The function $x(t)$, the sampling function $\Delta(t)$, and the windowing function $w(t)$ along with their Fourier transforms are shown in Figure 1.8.

Consider the collection of signal samples which fall in the data window: $x(k\Delta T)$ with $\{k = 0,1,2, \cdots, N-1\}$. These samples can be viewed as being obtained by the multiplication of the signal $x(t)$, the sampling function $\delta(t)$, and the windowing function $\omega(t)$:

$$y(t) = x(t)\delta(t)w(t) = \sum_{k=0}^{N-1} x(k\Delta T)\,\delta(t - k\Delta T)\,, \tag{1.14}$$

where once again the multiplication with the delta function is to be understood in the sense of the integral of Eq. (1.9). The Fourier transform of the sampled windowed function $y(t)$ is then the convolution of Fourier transforms of the three functions.

The Fourier transform of $y(t)$ is to be sampled in the frequency domain in order to obtain the DFT of $y(t)$. The discrete steps in the frequency domain are multiples of $1/T_0$, where T_0 is the span of the windowing function. The frequency sampling function $\Phi(f)$ is given by

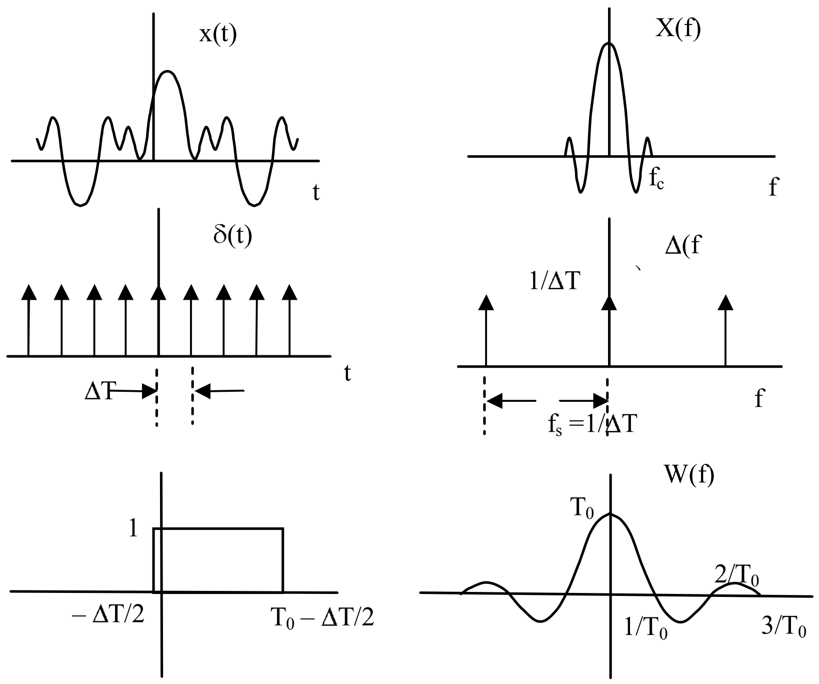

Fig. 1.8 Time functions and Fourier transforms $x(t)$, $\delta(t)$, and $\omega(t)$. Note that once again the phase shift factor from $\Omega(f)$ has been omitted.

$$\Phi(f) = \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T_0}) \tag{1.15}$$

and its inverse Fourier transform (by Property 6 of Fourier transforms) is

$$\phi(t) = T_0 \sum_{n=-\infty}^{\infty} \delta(t - nT_0). \tag{1.16}$$

In order to obtain the samples in the frequency domain, we must multiply the Fourier transform $Y(f)$ with $F(f)$. To obtain the corresponding time domain function $x'(t)$ we will require a convolution in the time domain of $y(t)$ and $\phi(t)$:

$$x'(t) = y(t) * \phi(t)$$

$$x'(t) = y(t) * \phi(t) = \left[\sum_{k=0}^{N-1} x(k\Delta T)\delta(t - k\Delta T) \right] * \left[T_0 \sum_{n=-\infty}^{\infty} \delta(t - nT_0) \right]$$

$$= T_0 \sum_{n=-\infty}^{\infty} \left[\sum_{k=0}^{N-1} x(k\Delta T)\delta(t - k\Delta T - nT_0) \right]. \tag{1.17}$$

This function is periodic with a period T_0. The functions $x(t)$, $y(t)$, and $x'(t)$ are shown in Figure 1.9. The windowing function limits the data to samples 0 through $N-1$, and the sampling in frequency domain transforms the original N samples in time domain to an infinite train of N samples with a period T_0 as shown in Figure 1.9 (c). Note that although the original function $x(t)$ was not periodic, the function $x'(t)$ is, and we may consider this function to be an approximation of $x(t)$.

The Fourier transform of the periodic function $x'(t)$ is a sequence of impulse functions in frequency domain by Property 5 of the Fourier transform. Thus

$$\begin{aligned} X'(f) &= \sum_{n=-\infty}^{\infty} \alpha_n\, \delta(f-\frac{n}{T_0}), \text{ with} \\ \alpha_n &= \frac{1}{T_0}\int_{-T_0/2}^{T_0-T_0/2} x'(t)\, e^{-\frac{j2\pi nt}{T_0}}\, \mathrm{d}t\,, \quad n = 0, \pm1, \pm2, \cdots. \end{aligned} \tag{1.18}$$

Substituting for $x'(t)$ in the above expression for α_n,

$$\begin{aligned} \alpha_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0-T_0/2}\left\{T_0\sum_{m=-\infty}^{\infty}\left[\sum_{k=0}^{N-1} x(k\Delta T)\,\delta(t-k\Delta T-mT_0)\right]\right\}e^{-\frac{j2\pi nt}{T_0}}\,\mathrm{d}t, \\ n = 0, \pm1, \pm2, \cdots \end{aligned} \tag{1.19}$$

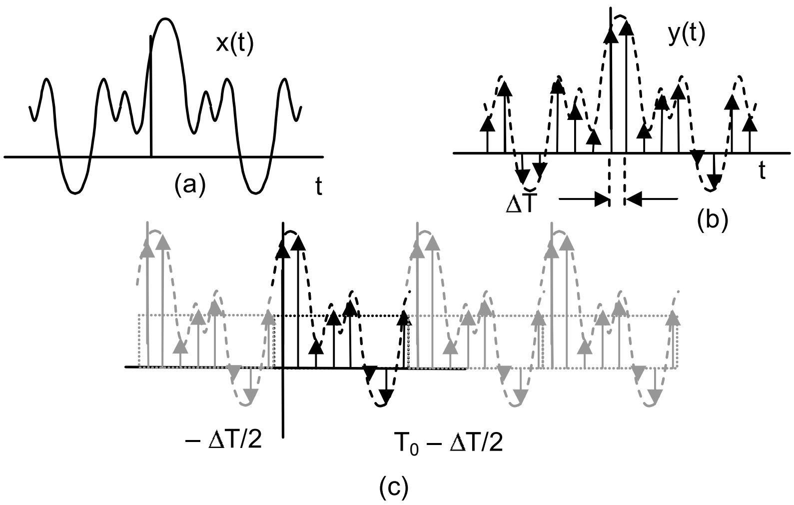

Fig. 1.9 (a) The input function $x(t)$, its samples (b), and (c) the Fourier transform of the windowed function $x'(t)$.

The index m designates the train of periods shown in Figure 1.9 (c). Since the limits on the integration span one period only, we may remove the summation on m, and set m = 0, thus using only the samples over the period shown in bold in Figure 1.9 (c). Equation (1.15) then becomes

$$\alpha_n = \int_{-T_0/2}^{T_0-T_0/2} \left[\sum_{k=0}^{N-1} x(k\Delta T)\,\delta(t-k\Delta T) \right] e^{-\frac{j2\pi nt}{T_0}}\,\mathrm{d}t\,, \quad or$$

$$\alpha_n = \sum_{k=0}^{N-1} \int_{-T_0/2}^{T_0-T_0/2} x(k\Delta T)\delta(t-k\Delta T)\,e^{-\frac{j2\pi nt}{T_0}}\,\mathrm{d}t, \quad = \sum_{k=0}^{N-1} x(k\Delta T)\,e^{-\frac{j2\pi nkn\Delta T}{T_0}} \tag{1.20}$$

$$n = 0, \pm 1, \pm 2, \cdots.$$

Since there are N samples in the data window T_0, $N\Delta T = T_0$. Therefore

$$\alpha_n = \sum_{k=0}^{N-1} x(k\Delta T)\,e^{-\frac{j2\pi kn}{N}}\,,\ with\ n = 0, \pm 1, \pm 2, \cdots. \tag{1.21}$$

Although the index n goes over all positive and negative integers, it should be noted that there are only N distinct coefficients α_n. Thus, α_{N+1} is the same as α_1 and the Fourier transform $X'(f)$ has only N distinct values corresponding to frequencies $f = n/T_0$, with n ranging from 0 through $N - 1$:

$$X'(\frac{n}{T_0}) = \sum_{k=0}^{N-1} x(k\Delta T)\,e^{-\frac{j2\pi kn}{N}}\,,\ with\ n = 0, 1, 2, \cdots N-1. \tag{1.22}$$

Equation (1.22) is the definition of the DFT of N input samples taken at intervals of ΔT. The DFT is symmetric about $N/2$, the components beyond $N/2$ simply belong to negative frequency. Thus the DFT does not calculate frequency components beyond $N/(2T_0)$, which also happens to be the Nyquist limit to avoid aliasing errors.

Also note that any real function of time can be written as a sum of a real and an odd function. Consequently, by Properties 2 and 3 above any real function of time will have real parts of the DFT as even functions of frequency and the imaginary parts of the DFT will be odd functions of frequency.

1.5.1 DFT and Fourier series

The Fourier series coefficients of a periodic signal can be obtained from the DFT of its sampled data by dividing the DFT by N, the number of samples in the data window. Thus, the Fourier series for a function $x(t)$ can be expressed by the formula

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{\frac{j2\pi kt}{T}} = \sum_{k=-\infty}^{\infty} \left[\frac{1}{N} \sum_{n=0}^{N-1} x(k\Delta T)\, e^{-\frac{j2\pi kn}{N}} \right] e^{\frac{j2\pi kt}{T}}. \tag{1.23}$$

As there are only N components in the DFT, the summation on k in Eq. (1.23) is from $\{k = 0, \cdots, N-1\}$.

Example 1.3

Consider a periodic function $x(t) = 1 + \cos 2\pi f_0 t + \sin 2\pi f_0 t$. The function is already expressed in terms of its Fourier series, with $a_0 = 2$, $a_1 = 1$, and $b_1 = 1$. The signal is sampled 16 times in one period of the fundamental frequency. The sampled data, the DFT, and the DFT divided by 16 (N, the number of samples) is shown in Table 1.1.

Table 1.1 Sampled data and Fourier transform of the periodic function $t = 1 + \cos 2\pi f_0 t + \sin 2\pi f_0 t$

Sample no.	$x(t)$	Frequency	DFT	**X** = DFT/16
0	2.0000	0	16.0000	1.000
1	2.3066	f_0	$8.0000 + j8.0000$	$0.5000 + j0.5000$
2	2.4142	$2f_0$	$0.0000 - j0.0000$	$0.0000 + j0.0000$
3	2.3066	$3f_0$	$0.0000 - j0.0000$	$0.0000 + j0.0000$
4	2.0000	$4f_0$	$0.0000 + j0.0000$	$0.0000 + j0.0000$
5	1.5412	$5f_0$	$-0.0000 + j0.0000$	$0.0000 + j0.0000$
6	1.0000	$6f_0$	$0.0000 + j0.0000$	$0.0000 + j0.0000$
7	0.4588	$7f_0$	$0.0000 - j0.0000$	$0.0000 + j0.0000$
8	0.0000	–	–0.0000	$0.0000 + j0.0000$
9	–0.3066	$-7f_0$	$0.0000 + j0.0000$	$0.0000 + j0.0000$
10	–0.4142	$-6f_0$	$0.0000 - j0.0000$	$0.0000 + j0.0000$
11	–0.3066	$-5f_0$	$-0.0000 - j0.0000$	$0.0000 + j0.0000$
12	–0.0000	$-4f_0$	$0.0000 + j0.0000$	$0.0000 + j0.0000$
13	0.4588	$-3f_0$	$0.0000 + j0.0000$	$0.0000 + j0.0000$
14	1.0000	$-2f_0$	$0.0000 + j0.0000$	0.0000 + j0.0000
15	1.5412	$-f_0$	$8.0000 - j8.0000$	$0.5000 - j0.5000$

The last column contains the Fourier series coefficients. Note that the DC component a_0 appears in the 0th position, while the fundamental frequency component appears in the 2nd and 15th position. The cosine term being an even function produces real parts which are even functions of frequency (0.5 at $\pm f_0$), while the sine term is an odd function of time and produces odd functions of frequency ($\pm j0.5$ at $\pm f_0$). The coefficient a_1 is obtained by adding the real parts corresponding to f_0 and $-f_0$ in the (DFT/16) column, while the coefficient b_1 is obtained by subtracting the imaginary part of the $-f_0$ term from the imaginary part of the f_0 term:

$a_0 = 2X_0 = 2$
$a_1 = \text{Real}(X_1 + X_{N-1}) = 1$
$b_1 = \text{Imaginary}(X_1 - X_{N-1}) = 1$

From the above example it is clear that for real functions $x(t)$ the Fourier series coefficients of a periodic function can be obtained from the DFT of its sampled data by the following formulas:

$a_0 = 2.X_0$
$a_k = 2.\text{Real}(X_k)$
$b_k = 2.\text{Imaginary}(X_k)$ for $k = 1, 2, ,\cdots, N/2 - 1$.

1.5.2 DFT and phasor representation

A sinusoid $x(t)$ with frequency kf_0 with a Fourier series

$$\begin{aligned} x(t) &= a_k \cos(2\pi k f_0 t) + b_k \sin(2k\pi f_0 t) \\ &= \left\{ \sqrt{(a_k^2 + b_k^2} \right\} \cos(2\pi k f_0 t + \phi) \quad \text{where } \phi = \arctan(\frac{-b_k}{a_k}) \end{aligned} \tag{1.24}$$

has a phasor representation (see Section 1.2)

$$X_k = \frac{1}{\sqrt{2}} \left\{ \sqrt{(a_k^2 + b_k^2} \right\} e^{j\phi}, \tag{1.25}$$

where the square-root of 2 in the denominator is to obtain the rms value of the sinusoid. The phasor in complex form becomes

$$X_k = \frac{1}{\sqrt{2}} (a_k - jb_k). \tag{1.26}$$

Using the relationship of the Fourier series coefficients with the DFT, the phasor representation of the kth harmonic component is given by

$$
\begin{aligned}
X_k &= \frac{1}{\sqrt{2}} \frac{2}{N} \sum_{n=0}^{N-1} x(n\Delta T)\, e^{-\frac{j2\pi kn}{N}} \\
&= \frac{\sqrt{2}}{N} \sum_{n=0}^{N-1} x(n\Delta T) \left\{ \cos\left(\frac{2\,\pi kn}{N}\right) - j \sin\left(\frac{2\pi kn}{N}\right) \right\}.
\end{aligned} \tag{1.27}
$$

Using the notation $x(n\Delta T) = x_n$, and $2\pi/N = \theta$ (θ is the sampling angle measured in terms of the period of the fundamental frequency component)

$$X_k = \frac{\sqrt{2}}{N} \sum_{n=0}^{N-1} x_n \left\{ \cos(kn\theta) - j \sin(kn\theta) \right\}. \tag{1.28}$$

If we define the cosine and sine sums as follows:

$$X_{kc} = \frac{\sqrt{2}}{N} \sum_{n=0}^{N-1} x_n \cos(kn\theta), \tag{1.29}$$

$$X_{ks} = \frac{\sqrt{2}}{N} \sum_{n=0}^{N-1} x_n \sin(kn\theta), \tag{1.30}$$

then the phasor X_k is given by

$$X_k = X_{kc} - jX_{ks}. \tag{1.31}$$

Equations (1.29) through (1.31) will be used to represent the phasor in most of the computations in the rest of our discussion.

Example 1.4

Consider a signal consisting of a DC component, and 60 Hz, 120 Hz, and 300 Hz components:

$x(t) = 0.5 + \cos(120\pi t + \pi/4) + 0.2 \cos(240\pi t + \pi/8) + 0.3 \cos(600\pi t)$.

Note that the signal is real, but not an even or odd function of time, and hence by Property 4 above, the real part of the Fourier transform will be even, and the imaginary part will be odd functions of frequency.

The signal is sampled at 1440 Hz, and the following 24 samples are obtained over a window of 16.66 ms, which corresponds to one period of the 60-Hz signal. There will be 24 frequency samples of the DFT. They are calculated and tabulated in Table 1.2.:

Table 1.2 Spectrum created by DFT

Sample no.	$x(k)$	Frequency	DFT	DFT/24
0	1.6919	0	$12.0000 + j0.0000$	$0.5000 + j0.0000$
1	1.1994	f_0	$8.4853 - j8.4853$	$0.3535 - j0.3535$
2	0.5251	$2f_0$	$2.2173 - j0.9184$	$0.0924 - j0.0383$
3	0.2113	$3f_0$	$0.0000 - j0.0000$	$0.0000 - j0.0000$
4	0.2325	$4f_0$	$0.0000 - j0.0000$	$0.0000 - j0.0000$
5	0.0915	$5f_0$	$3.6000 - j0.0000$	$0.1500 - j0.0000$
6	–0.3919	$6f_0$	$-0.0000 - j0.0000$	$-0.0000 - j0.0000$
7	–0.7776	$7f_0$	$0.0000 - j0.0000$	$0.0000 - j0.0000$
8	–0.6420	$8f_0$	$-0.0000 - j0.0000$	$-0.0000 - j0.0000$
9	–0.2113	$9f_0$	$-0.0000 - j0.0000$	$-0.0000 - j0.0000$
10	–0.0474	$10f_0$	$-0.0000 + j0.0000$	$-0.0000 + j0.0000$
11	–0.2454	$11f_0$	$0.0000 - j0.0000$	$0.0000 - j0.0000$
12	–0.3223	–	$0.0000+ j0.0000$	$0.0000 + j0.0000$
13	0.0441	$-11f_0$	$0.0000 + j0.0000$	$0.0000 + j0.0000$
14	0.5271	$-10f_0$	$-0.0000 - j0.0000$	$-0.0000 - j0.0000$
15	0.6356	$-9f_0$	$-0.0000 + j0.0000$	$-0.0000 + j0.0000$
16	0.4501	$-8f_0$	$-0.0000 + j0.0000$	$-0.0000 + j0.0000$
17	0.5119	$-7f_0$	$0.0000 + j0.0000$	$0.0000 + j0.0000$
18	1.0223	$-6f_0$	$-0.0000 + j0.0000$	$-0.0000 + j\,0.0000$
19	1.5341	$-5f_0$	$3.6000 + j0.0000$	$0.1500 + j\,0.0000$
20	1.5898	$-4f_0$	$0.0000 + j0.0000$	$0.0000 + j\,0.0000$
21	1.3644	$-3f_0$	$0.0000 + j0.0000$	$0.0000 + j\,0.0000$
22	1.3648	$-2f_0$	$2.2173 + j0.9184$	$0.0924 + j\,0.0383$
23	1.6420	$-f_0$	$8.4853 + j8.4853$	$0.3535 + j\,0.3535$

The Fourier series coefficients are

$a_0 = 1.0$,
$a_1 = 0.707$,
$b_1 = -0.707$,
$a_2 = 0.1848$,
$b_2 = -0.0766$,
$a_5 = 0.3$,
$b_5 = 0.000$,

leading to the Fourier series

$x(t) = 0.5 + 0.707\cos(120\pi t) - 0.707\sin(120\pi t) + 0.1848\cos(240\pi t) - 0.0766\sin(240\pi t) + 0.3\cos(600\pi t)$

which agrees with the expression for the input signal.