monorail, deformation, crew, mathematical model, carriage

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A MATHEMATICAL MODEL STUDY OF SUSPENDED MONORAIL

Summary. The mathematical model of suspended monorail track with allowance for elastic strain which occurs during movement of the monorail carriage was developed. Standard forms for single span and double span of suspended monorail sections were established.

МАТЕМАТИЧЕСКАЯ МОДЕЛЬ ПОДВЕСНОГО МОНОРЕЛЬСОВОГО ПУТИ

Аннотация. Разработана математическая модель подвесного пути монорельсовой дороги, учитывающая упругие деформации монорельса, происходящие во время движения экипажа. Установлены нормальные формы для одиночного и двойного пролета секций подвесного монорельсового пути.

1. INTRODUCTION

Monorail tracks belong to the transport that makes possible to move materials and people along the lines with the alternating profile. The feature that differs this means of transport from the other ones is the gravity centre of rolling stock which is located below the monorail track suspended on the high stocks, trestles or the mine timbering.

Monorail track consists of sections. Each section has fastenings for suspending and for binding with each other. On straight lines a section has the length of up to 3 m and is normally fastened at the edge points from both sides. At the curved lines there are normally used either sections to the length of which is equal to 1 m or sections which are manufactured curved; that depends on the degree of curvature. These sections are suspended at least in three points.

By moving along the track a monorail causes additional dynamic loadings which, via the suspension arrangement, affect the mine timbering cutting its stability. That’s why it is a vital task for studying the process of interaction between the carriage and the monorail track.

Scientific researches in the sphere of rail transport [1, 2, 3] deal with the dynamic processes which occur while moving of carriage along the track that is built on the ground. The specific features of the suspended monorail track and the peculiarities of its usage makes it impossible to apply the earlier gained results.

The goal of this article is to define the interconnection between performance characteristics of monorail track and its suspension. The calculation is provided under the following assumption: the elastic stiffness is constant along the monorail track; the side sway is eliminated by using tie-rods; the contact of carriage with the track is fixed and mechanical characteristics are constant.

2. RESERCH AND RESULTS

The suspension of monorail track is an elastic structure, that's why its mathematical model should be developed with allowance for elastic strain that occurs during the carriage moving. If we include interaction forces and implement the principal of removing constrains, the model «monorail-carriage» can be rendered as a periodical structure with the consequent elements at the suspended supports (Fig. 1).

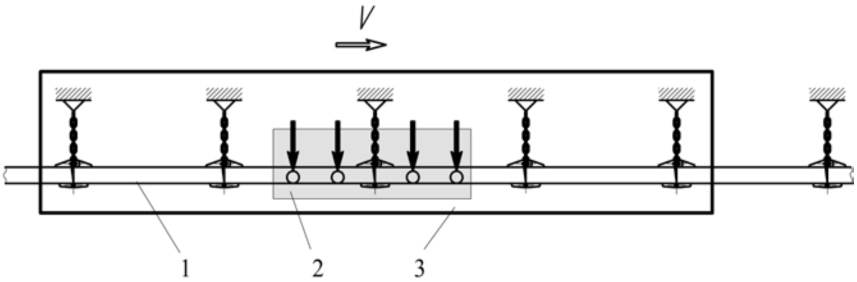

Fig. 1. Diagram of suspended monorail carriage model: 1 – monorail; 2 – carriage; 3 – system
Рис. 1. Схема модели подвесной монорельс-экипаж: 1 – монорельс; 2 – экипаж; 3 – система

The number of interacting elements is determined by the length of sections and depends on the quantity of moving carriages and their length. Each section can be presented as a multisupported beam with the length L, consisting of ξ similar segments of the length L_i, constant mass equal to q_i and beam stiffness to EI (Fig. 2). By carriage moving with the speed V causes the power that affects the monorail $P_\mu(t), \mu = 1,...m,$ which result to segment strain with bending $y_i(\xi_i, t)$.

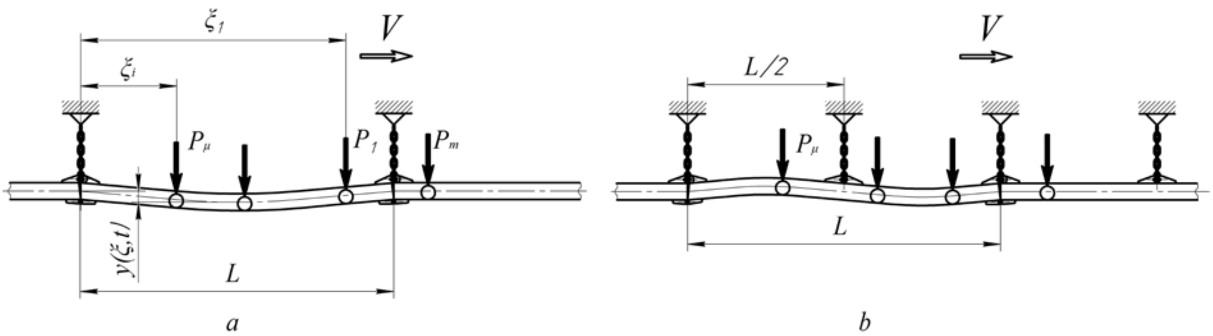

Fig. 2. Diagram of monorail track strain caused by the moving forces for spans: *a* – single; *b* – double
Рис. 2. Схема деформации монорельсового пути под движущимися силами для пролетов: *a* – одиночного; *b* – двойного

These bends can be detected with the help of Bernoulli-Euler small deformations theory [4, 5]

$$(EI)_i y_i^{''''}(\xi_i, t) + q_i y_i^{''}(\xi_i, t) = \sum_\mu P_\mu(t)\delta(\xi_i - \xi_{i\mu}), \tag{1}$$

where: i = 1, 2, 3, …s; δ – Dirac function; $\xi_{i\mu}$ – force coordinate $P_\mu(t)$, that affects i-segment (monorail section).

The equation can be solved by using initial and boundary conditions

$$y_i(\xi_i, t) = \sum_{j=1}^{\infty} \varphi_{ij}(\xi_i) z_j(t), \tag{2}$$

where: i = 1, 2, 3, …s; $\varphi_{ij}(\xi_i)$ – normal form; $z_j(t)$ – generalized coordinate.

Normal forms are eigenvalue equation solution

$$\varphi_i^{''''}(\xi_i) - \lambda_i^4 \varphi_i(\xi_i) = 0, \tag{3}$$

where: λ – is proper system;

$$\lambda_i^4 = y^2 (q_i / EI)_i .$$

This equation describes proper frequency ω_j and the corresponding proper frequencies $\varphi_{ij}(\xi_i)$, that satisfies the boundary conditions and are orthogonal to monorail.

By $j \neq k$

$$\sum_{i=1}^{s} \int_0^{L_i} q_i \varphi_{ij}(\xi_i) \varphi_{ik}(\xi_i) d\xi_i = 0. \tag{4}$$

If $j=k$

$$\sum_{i=1}^{s} \int_0^{L_i} q_i \varphi_{ij}(\xi_i) \varphi_{ik}(\xi_i) d\xi_i = M_j = \sum_{i=1}^{s} M_{ij}. \tag{5}$$

It follows that

$$\ddot{z}_j(t) + \omega_j^2 z_j(t) = \frac{1}{M_{i\mu}} \sum \varphi_{ij}(\xi_{i\mu}) P_\mu(t) . \tag{6}$$

For single span of monorail track shown in Fig. 2*a*, standard forms can be formulated as follows

$$\varphi_i\ (\xi_i) = C_1 \cosh \lambda\xi / L_i + C_2 \sinh \lambda\xi / L_i + C_3 \cos \lambda\xi / L_i + C_4 \sin \lambda\xi / L_i. \tag{7}$$

Using vector representation $\varphi_i\ (\xi_i) = a^T(\lambda\xi / L_i)c,$

where: $a(\cdot) = [C(\cdot), S(\cdot), s(\cdot)]^T$; $c = [C_1, C_2, C_3, C_4]^T$; $C(\cdot) = \cosh(\cdot)$; $c(\cdot) = \cos(\cdot)$; $S(\cdot) = \sinh(\cdot)$; $s(\cdot) = \sin(\cdot)$

and boundary conditions $\varphi_i(0) = 0; \varphi_i^{''}(0) = 0; \varphi_i(L_i) = 0; \varphi_i^{''}(L_i) = 0,$

when $C_1 = C_2 = C_3 = 0$ and $C_4 \sin \lambda = 0,$ we have the following solution $\lambda_j = \pi j,$

$$\omega_j^2 = \left(\frac{\pi j}{L_i}\right)^4 \frac{EI}{q_i}, \quad j = 1, 2, 3 ... \tag{8}$$

In case, when $C_4 = 1,$

$$\varphi_i = \sin \pi j, M_j = \int_0^{L_i} q_i \varphi_j^2(\xi) d\xi = \frac{q_i L}{2}. \tag{9}$$

Shall we make analysis for double span monorail shown in Fig. 2*b*, using the finite elements method [4].

Monorail track is divided into s segments; let us define common and local coordinates for them. We add powers Q_l , Q_h and moments M_l , M_h., imposed at the edge points of each segment. Junction turns caused by the load we shall denote as ξ_l and ξ_h, and their shifts as y_l and y_h (Fig. 3*a*).

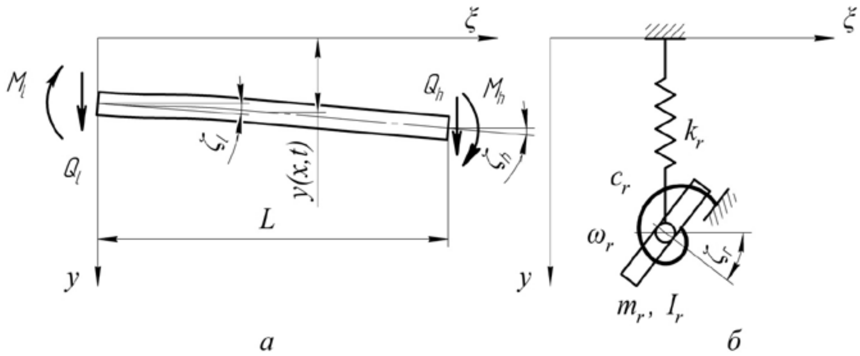

Fig. 3. Diagram of monorail strain: *a* – *s* segment; *b* – support fastening assembly
Рис. 3. Схема деформации монорельса: *а* – *s* сегмента; *b* – узла крепления опор

The vector mode of the load will be the following $f = [M_l Q_l M_h Q_h]^T$.

By analogy the segment strain will be $v = [\xi_l y_l \xi_h y_h]^T$.

In the form of vector mode

$$f = EI[-\varphi''(0), \varphi'''(0), \varphi''(L), -\varphi'''(L)]^T = D(\lambda)c, \tag{10}$$

$$v = [-\varphi'(0), \varphi(0), \varphi'(L), -\varphi(L)]^T = C^{-1}(\lambda)c. \tag{11}$$

With the allowance that in the beginning of the segment $\xi_l = 0$, and at the end – $\xi_h = L$.

Regarding matrices $D(\lambda)$ and $C^{-1}(\lambda)$

$$f = vD(\lambda)C(\lambda) = vP(\lambda), \tag{12}$$

where: $P(\lambda)$ – matrix of segment dynamic stiffness;

$$P(\lambda) = EI\begin{bmatrix} \dfrac{P_2}{L} & -\dfrac{P_4}{L^2} & \dfrac{P_1}{L} & -\dfrac{P_3}{L^2} \\ -\dfrac{P_4}{L^2} & \dfrac{P_6}{L^3} & \dfrac{P_3}{L^2} & \dfrac{P_5}{L^2} \\ \dfrac{P_1}{L} & \dfrac{P_3}{L^2} & \dfrac{P_2}{L} & \dfrac{P_4}{L^2} \\ -\dfrac{P_3}{L^2} & \dfrac{P_5}{L^3} & \dfrac{P_4}{L^2} & \dfrac{P_6}{L^3} \end{bmatrix};$$

$$P_1 = -\frac{\lambda\big(S(\lambda) - s(\lambda)\big)}{N};\; P_2 = -\frac{\lambda\big(C(\lambda)s(\lambda) - S(\lambda)c(\lambda)\big)}{N};\; P_3 = -\frac{\lambda^2\big(C(\lambda) - c(\lambda)\big)}{N};$$

$$P_4 = \frac{\lambda^2\big(S(\lambda)s(\lambda)\big)}{N};\; P_5 = \frac{\lambda^2\big(S(\lambda) + s(\lambda)\big)}{N};\; P_6 = -\frac{\lambda^2\big(C(\lambda)s(\lambda) + S(\lambda)c(\lambda)\big)}{N};$$

$$N = C(\lambda)c(\lambda) - 1.$$

The monorail track, divided into *s* segments, includes *k* junctions and can be presented as follows

$$\lambda_i = \lambda_i(\omega) = L\sqrt[4]{\omega^2\left(\frac{q_i}{EI}\right)_i};$$
$$\varphi_i = \varphi_i(\xi_i) = a_i^T c_i;\ c_i = C_i v_i;\ f_i = P_i v_i; \tag{13}$$
$$a_i = a(\lambda_i \xi_i / L_i); C_i = C\lambda_i; P_i = P(\lambda_i); i = 1,2,3\ldots s.$$

By means of generalized coordinates b_v, where v =1, 2, 3,…n,

$$b = [b_1,\ldots b_n]^T;$$
$$v_g = [v_1^T,\ldots v_n^T]^T; \tag{14}$$
$$P_g = diag(P_i), i = 1,2,3,\ldots s.$$

In the matrix form $v_i = I_i b, v_g = I_g b.$

Therefrom applying the virtual work principle,

$$\delta v_g = \sum_{i=1}^{s} \delta v_i^T f_i = \sum_{i=1}^{s} \delta v_i^T P_i v_i = \delta v_i^T P_g v_g = \delta b^T I_g^T P_g I_g b = 0. \tag{15}$$

For free variations δb, the following is valid

$$I_g^T P_g(\omega) I_g b = \overline{P}(\omega) b = 0, \tag{16}$$

where $\overline{P}(\omega)$ – dynamic matrix of segments stiffness.

Shall we describe the influence of monorail track supports yield (Fig. 3*b*). It can be noted by means of virtual work of the applied forces that can be defined according to the following

$$\delta W_g = -\sum_{r=1}^{k} [\delta\xi_r (c_r - \omega^2 J_r)\xi_r + \delta\omega_r (k_r - \omega^2 m)\omega_r] = -\delta b^T (K^Z - \omega^2 M^Z) b, \tag{17}$$

where: c_r, k_r – stiffness factor; m, J_r – element mass and moment of inertia of; K^Z – elasticity matrix; M^Z – moment of inertia matrix.

Regarding that $\delta v_g = \delta W_g$, we have

$$[\overline{P}(\omega) + K^Z - \omega^2 M^Z] b = \hat{P}(\omega) b, \tag{18}$$

where $\hat{P}(\omega)$ – dynamic full stiffness matrix.

This equation solution gives an opportunity to set proper frequencies ω_j and proper vectors b_j with j=1, 2, 3,…f.

Functions $\varphi_{ij}(\xi_i)$ can be determined according to the following

$$\lambda_i = \lambda_i(\omega_j) = L_j \sqrt[4]{\omega_j^2\left(\frac{q_i}{EI}\right)_i}; \tag{19}$$
$$v_{ij} = I_i b_j;\ c_{ij} = C_{ij} v_{ij};\ C_{ij} = C_i \lambda_{ij};\ \varphi_{ij} = a_{ij}^T c_{ij} = a_{ij}^T C_{ij} v_{ij},\ a_{ij} = a_i(\lambda_{ij}\xi_i / L_i).$$

The results of modal analysis for double-span monorail section of suspended tracks [2, 3] are shown in Fig. 4.

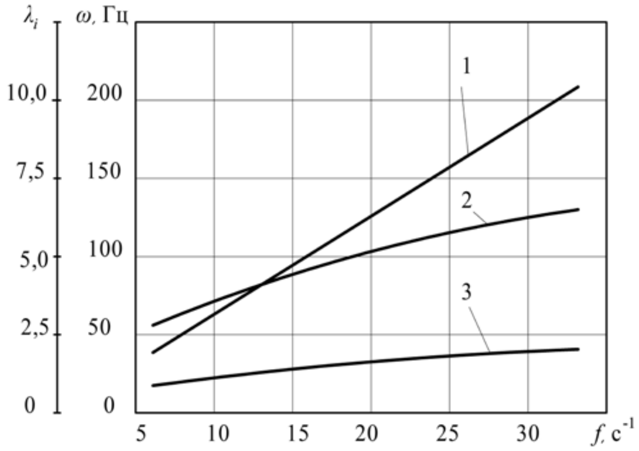

Fig. 4. Dependences: 1– $\omega(f)$; 2 – $\lambda_1(f)$ if L_1= 6 м; 3 – $\lambda_2(f)$ and L_2= 3 м
Рис. 4. Зависимости: 1– $\omega(f)$; 2 – $\lambda_1(f)$ при L_1= 6 м; 3 – $\lambda_2(f)$ при L_2= 3 м

As it is shown in Fig. 4 the frequency $\omega(f)$ increases unlimited with the increasing of f. And the fixing of k-junction practically doesn't influence it. The increase of f causes the increasing of $\lambda_1(f)$, but with f more than 30 c^{-1} it reaches its maximum. If section length L_1 decreases half as much, $\lambda_1(f)$ goes down more than twice and a half.

It's necessary to highlight, that removal of k-junction fixing causes the increase of $\lambda_1(f)$ by 3...5%, that's why it is not obvious to consider it while engineering design.

3. CONCLUSIONS

The mathematical model of suspended monorail track with allowance for elastic strain which occurs during movement of the monorail carriage was developed with the help of virtual work principle. Standard forms for single span and double span monorail sections were established. The carried out modal analysis of monorail path shows that with the increase of section length from 3 m up to 6 m, index f increases not less than to 2,5 times. Increasing f causes permanent increase of frequency $\omega(f)$, and the additional fixing of suspension junction doesn't change this reaction.

The further researches should aim the developing of dependences for defining the optimal characteristics of monorail tracks with allowance for variable character of loads to the monorail.

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