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Stabilization of Networked Control Systems with Data Packet Dropout and Transmission Delays: Continuous-Time Case*

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The problem of data packet dropout and transmission delays induced by communication channel in networked control systems (NCSs) is studied in this paper. We model the continuous-time NCSs with data packet dropout and transmission delays as ordinary linear systems with time-varying input delays. By using the Lyapunov–Razumikhin function techniques, delay-dependent condition on the stabilization of NCSs is obtained in terms of linear matrix inequalities (LMIs). Stabilizing state feedback controllers can then be constructed by using the feasible solutions of some LMIs. The admissible upper bounds of data packet loss and delays can be computed by using the quasi-convex optimization algorithm. Numerical examples illustrate the effectiveness of the proposed approach.

Keywords: Data Packet Dropout; Delay Systems; Linear Matrix Inequalities; Networked Control Systems; Stabilization; Transmission Delays

1. Introduction

Networks have received increasing attention in recent years due to the popularization and advantages of using network cables in control systems [12,19]. Systems where feedback control loops are closed through a real-time network are often referred to as networked control systems (NCSs). The system elements are typically spatially isolated from one another, operating in an asynchronous manner and communicating over a wide area via both wired and wireless links. Advantages of NCSs include low cost, high reliability, less wiring and easy maintenance, etc. Typical examples are distributed industrial control/automation, intelligent traffic systems, satellite clusters and group maneuvers, mobile sensor arrays, multiple autonomous mobile robots, large-scale decentralized flexible manufacturing systems, formation of unmanned air vehicles, multi-agent systems, and advanced aircraft and spacecraft networks, etc. However, the insertion of communication networks in the feedback control loop complicates the application of standard results in analysis and design of NCSs because many ideal assumptions made in the traditional control theory can not be applied to NCSs directly (see, e.g. [17–19] and the references therein).

One of the key issues arising in NCSs is the unreliable transmission paths because of limited bandwidth and large amount of data packet transmitted over single channel. Data packet dropout often occurs while exchanging data among devices such as sensors, actuators, and controllers, and this can degrade performance and destabilize the system. An augmented state space method is developed to deal with the problem of data packet dropout [19]. The

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performance of real-time NCSs with data dropout is discussed in [10]. Using an uncertainty threshold principle, Azimi-Sadjadi [1] presents a general framework for stability analysis of NCSs. However, the research mentioned above is concerned primarily with analysis issues rather than control design.

Since the network is tied with control systems, network-induced delays are inherent to NCSs and thus always affect the performance of NCSs. So far, various methodologies have been proposed to deal with the problem of network delays. An augmented state vector method is proposed in [16] to control a linear system over a periodic delay network. Queuing mechanisms are developed in [4,11], which utilize some deterministic or probabilistic information of NCSs for control purpose. Random delays are discussed in [14] via an optimal stochastic control methodology, see also [3,9] and the references therein for related works. However, no method has been given in the references mentioned above on how to estimate the maximum allowable value of the network-induced delays that preserves the stability of NCSs.

Because data packet dropout and transmission delays might be potential sources to the instability and poor performance of NCSs, this paper considers stabilization of such NCSs. The NCSs with data packet dropout and transmission delays are modelled as linear systems with time-varying input delay, which might be subject to fast time-varying. In the literature, there are two main approaches to dealing with linear systems with time-varying delay: one is the Lyapunov–Krasovskii functional method and the other is the Lyapunov–Razumikhin function method. Usually, Lyapunov–Krasovskii functional method requires the varying delay to satisfy $\dot{\tau}(t) < 1$, where $\tau(t)$ is the delay function. Unfortunately, the delay $\tau(t)$ modelled from NCSs in this paper does not satisfy this condition. For the Lyapunov–Razumikhin function method, though there are some results on the design of state feedback controller to stabilize a linear system with time-varying input delay [13], the controller can only be constructed via solving LMIs problem and quasi-convex optimization problem iteratively, which is very complex. By using certain inequality condition, this paper constructs the feedback controller directly via solving a set of LMIs, and thus the controller can be easily obtained. Furthermore, the admissible upper bound of data packet loss and delays can be obtained by using the quasi-convex optimization algorithm.

The paper is organized as follows. Section 2 models an NCS with data packet dropout as a linear system with time-varying input delay. Section 3 develops sufficient conditions on the stabilization of such NCSs. Moreover, the desired state feedback controller can be constructed in terms of LMIs. Section 4 deals with the problem of data packet dropout and transmission delays in a similar manner. Section 5 derives sufficient conditions on the stabilization of NCSs with multiple-packet transmission. Section 6 presents numerical examples to illustrate the efficiency and feasibility of our proposed approach. Section 7 concludes this paper.

2. Model of an NCS with Data Packet Dropout

In this section, single packet transmission is considered, where all the sensor data are lumped together into one network packet and transmitted at the same time. We assume that the actuator and the sensor used to measure the process output are connected through a communication channel with finite bandwidth, which is shared by several NCSs. Data packet dropout in an NCS is unavoidable because of limited bandwidth and several NCSs competing for one network channel. When packet collision occurs, it might be more advantageous to drop the old packet and transmit a new one than repeated retransmission attempt. An NCS with the possibility of dropping data packet can be described as in Fig. 1. The model consists of a continuous-time plant

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

and a piecewise continuous controller (realized by a zero-order-hold (ZOH))

$$u(t) = F\bar{x}(t), \quad t \in [t_k, t_{k-1}), \ \ k = 1, 2, \ldots, \tag{2}$$

where $x(t) \in \mathbf{R}^n, u(t) \in \mathbf{R}^m$ are the plant state and the plant input, respectively. F is the state feedback gain matrix to be designed, A, B are known real constant matrices of appropriate dimensions. We assume that the pair (A, B) is stabilizable. The sensor shown in

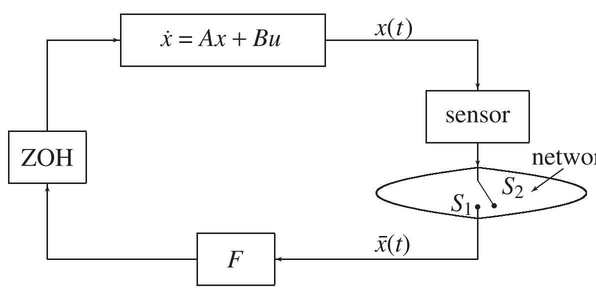

Fig. 1. An NCS with data packet dropout.

Fig. 1 takes on the work of sampling and transmission. The sampling period is a fixed positive constant scalar h, t_k is the sampling instant, and $\bar{x}(t)$ is the dynamics of the network.

We first consider the case where the controller and the actuator are combined into one node and there are no transmission delays between the sensor and the combined node. The network is modelled as a switch. When the switch is closed (in position S_1), the data packet containing $x(kh)$ is transmitted, and the controller utilizes the updated data; whereas when it is open (in position S_2), the output of the switch is held at the previous value, the data packet is lost, and the controller uses the old data. For a fixed sampling period, what we are interested in is the maximum quantity of packet loss that does not destabilize the closed-loop system. The dynamics of the switch can be expressed as follows:

the NCS (1)–(2) with no packet dropout at time $t_k : \bar{x}(t) = x(t_k)$;
the NCS (1)–(2) with one packet dropout at time $t_k : \bar{x}(t) = x(t_k - h)$;

$\vdots$

the NCS (1)–(2) with $d(k)$ packet dropout at time $t_k : \bar{x}(t) = x(t_k - d(k)h)$.

The quantity of dropped packets is accumulated from the latest time when $\bar{x}(t)$ has been updated. Thus the closed-loop system with the effect of packet loss is described as

$$\dot{x}(t) = Ax(t) + BFx(t_k - d(k)h), \quad t \in [t_k, t_{k+1}).$$

Let $\tau(t) = t - t_k + d(k)h$, then the system can be expressed as:

$$\dot{x}(t) = Ax(t) + BFx(t - \tau(t)), \quad t \in [t_k, t_{k+1}). \tag{3}$$

The quantity $d(k) \in Z^+$ may vary with time t and it is assumed that

$$0 \le d(k) \le d_k < \infty$$

for some positive integer d_k. Hence the delay function $\tau(t)$ satisfies

$$0 \le \tau(t) = t - t_k + d(k)h \le (\bar{d} + 1)h, \quad t \in [t_k, t_{k+1}),$$

where $\bar{d} = \max_k\{d_k\} < \infty$. Also it is assumed that $(\bar{d} + 1)h \le \bar{\tau}$, where $\bar{\tau}$ is a positive scalar.

In this way, we can model the NCS (1)–(2) with the effect of data packet dropout as a linear delay system (3), and this allows for the use of delay system theory to study the NCS under consideration.

Note that the time delay in system (3) is time-varying, and does not satisfy the condition: $\dot{\tau}(t) < 1$, as is usually assumed by the Lyapunov–Krasovskii method [8]. Hence we will employ the Lyapunov–Razumikhin method to study NCSs in the sequel.

3. Stabilization of NCSs with Data Packet Dropout

The following lemma will be used in the proof of our main results.

Lemma 1 ([15]). For any positive definite matrix $Q \in \mathbf{R}^{n \times n}$, the following inequality holds:

$$2x^{\mathrm{T}}y \le x^{\mathrm{T}}Q^{-1}x + y^{\mathrm{T}}Qy,$$

where $x \in \mathbf{R}^n, y \in \mathbf{R}^n$.

Sufficient condition on the stabilization of the NCS (1)–(2) with data packet dropout is developed as follows:

Theorem 1. Given an integer $\bar{\tau} > 0$, if there exist symmetric positive definite matrices $X, R_1, R_2 \in \mathbf{R}^{n \times n}$, a symmetric positive definite matrix $Q \in \mathbf{R}^{m \times m}$, a matrix $Y \in \mathbf{R}^{m \times n}$ satisfying the following LMIs:

$$\begin{bmatrix} \frac{1}{\bar{\tau}}(XA^{\mathrm{T}} + AX + Y^{\mathrm{T}}B^{\mathrm{T}} + BY) + 2X & BQ \\ QB^{\mathrm{T}} & -Q \end{bmatrix} < 0, \tag{4}$$

$$\begin{bmatrix} -X & Y^{\mathrm{T}}B^{\mathrm{T}} \\ BY & -R_2 \end{bmatrix} \le 0, \tag{5}$$

$$\begin{bmatrix} -X & XA^{\mathrm{T}} \\ AX & -R_1 \end{bmatrix} \le 0, \tag{6}$$

$$\begin{bmatrix} -Q & Y \\ Y^{\mathrm{T}} & -X \end{bmatrix} \le 0, \tag{7}$$

$$R_1 + R_2 - X \le 0, \tag{8}$$

then the NCS (1)–(2) is asymptotically stable with

$$F = YX^{-1}$$

for data packet dropout satisfying $0 \le d_k \le (\bar{\tau}/h) - 1$.

Proof. From the previous section, the NCS (1)–(2) with data packet dropout effect can be modelled as linear delay system (3). Hence, we only need to consider system (3). In each piecewise continuous interval of $\tau(t)$, we apply the Newton–Leibniz formula to system (3) and get

$$\dot{x} = (A + BF)x(t) - BF\int_{t-\tau(t)}^{t} (Ax(\theta) + BFx(\theta - \tau(\theta)))\,\mathrm{d}\theta \quad (9)$$

with the initial condition $x(t_0 + \theta) = \phi(\theta)$ for $\theta \in \xi_{t_0, 2\tau}$, where ϕ is a continuous norm-bounded initial function and $\xi_{t_0, 2\tau} = \{t \in R : t = \eta - 2\tau(\eta) \leq t_0, \eta \geq t_0\}$. As shown in [7], the asymptotic stability of system (3) can be guaranteed by that of system (9). Thus we proceed to analyze the latter.

Consider the following Lyapunov function

$$V(x(t)) = x^{\mathrm{T}}(t)Px(t),$$

where P is a symmetric positive definite matrix. The time derivative of $V(x(t))$ along the trajectory of system (9) is given by

$$\dot{V}(x(t)) = x^{\mathrm{T}}(t)(P(A + BF) + (A + BF)^{\mathrm{T}}P)x(t) - 2x^{\mathrm{T}}(t)PBF\int_{t-\tau(t)}^{t}(Ax(\theta) + BFx(\theta - \tau(\theta)))\,\mathrm{d}\theta. \quad (10)$$

Let $X = P^{-1}, F = YP$. Pre- and post-multiplying (5) and (6) by block-diag$\{P, P\}$, using the standard Schur complement, we can get the following inequalities:

$$A^{\mathrm{T}}R_1^{-1}A \leq P, \quad (BF)^{\mathrm{T}}R_2^{-1}BF \leq P. \quad (11)$$

To employ the Razumikhin-type theorem [7], we evaluate $\dot{V}(x(t))$ for the case

$$V(x(\theta)) < \delta V(x(t)), \quad t - 2\tau \leq \theta \leq t, \quad (12)$$

with $\delta > 1$. From (11)–(12) and Lemma 1, it follows that

$$\begin{aligned} &-2x^{T}(t)PBF\int_{t-\tau(t)}^{t} Ax(\theta)\,\mathrm{d}\theta \\ &\leq \int_{t-\tau(t)}^{t} x^{\mathrm{T}}(\theta)A^{\mathrm{T}}R_1^{-1}Ax(\theta)\,\mathrm{d}\theta + \tau(t)x^{\mathrm{T}}(t)PBFR_1F^{\mathrm{T}}B^{\mathrm{T}}Px(t) \\ &\leq \int_{t-\tau(t)}^{t} x^{\mathrm{T}}(\theta)Px(\theta)\,\mathrm{d}\theta + \tau(t)x^{\mathrm{T}}(t)PBFR_1F^{\mathrm{T}}B^{\mathrm{T}}Px(t) \\ &\leq x^{\mathrm{T}}(t)\tau(t)(\delta P + PBFR_1F^{\mathrm{T}}B^{\mathrm{T}}P)x(t) \end{aligned} \quad (13)$$

$$-2x^{\mathrm{T}}(t)PBF\int_{t-\tau(t)}^{t} BFx(\theta - \tau(\theta))\,\mathrm{d}\theta \leq x^{\mathrm{T}}(t)\tau(t)(\delta P + PBFR_2F^{\mathrm{T}}B^{\mathrm{T}}P)x(t). \quad (14)$$

Inserting (13) and (14) into (10) yields

$$\dot{V}(x(t)) \leq x^{\mathrm{T}}(t)(P(A + BF) + (A + BF)^{\mathrm{T}}P + \tau(t)(2\delta P + PBF(R_1 + R_2)F^{\mathrm{T}}B^{\mathrm{T}}P))x(t).$$

Note that the LMIs (7) and (8) can be easily transformed into the following inequalities,

$$P^{-1} - (R_1 + R_2) \geq 0, \quad Q - FP^{-1}F^{\mathrm{T}} \geq 0.$$

Hence $\dot{V}(x(t)) < 0$ for $V(x(\theta)) < \delta V(x(t))$ $(t - 2\tau \leq \theta \leq t)$ if

$$P(A + BF) + (A + BF)^{\mathrm{T}}P + \tau(t)(2\delta P + PBQB^{\mathrm{T}}P) < 0. \quad (15)$$

Pre- and post-multiplying (4) by block-diag$\{P, I\}$, and using Schur complement, we can get that (4) is equivalent to

$$P(A + BF) + (A + BF)^{\mathrm{T}}P + \bar{\tau}(2P + PBQB^{\mathrm{T}}P) < 0.$$

By the continuity of (15) in δ, (4) guarantees that there exists a $\delta > 1$ sufficiently small such that (15) holds for $\tau(t) \leq \bar{\tau}$. Furthermore, we have $F = YX^{-1}$. This completes the proof. □

Remark 1. Theorem 1 provides a method of designing a state feedback controller to stabilize an NCS with data packet loss. One salient feature of Theorem 1 is that the upper bound of $\tau(t)$, which is related to the allowable data packet loss that does not destabilize the NCS, can be calculated directly by solving the following quasi-convex optimization problem:

$$\begin{aligned} &\text{maximize } \bar{\tau}, \\ &\text{subject to } \quad \exists X > 0, R_1 > 0, R_2 > 0, \\ &\qquad Q > 0, Y \text{ satisfying (4)–(8).} \end{aligned} \quad (16)$$

Remark 2. Based on Lyapunov–Krasovskii method, stability analysis for linear system with time-varying state delay was conducted in [5], where there is no restriction on the derivative of the delay. However, their results can not be directly used to design stabilizing controller for system (3).

4. Stabilization of NCSs with Transmission Delay and Data Packet Dropout

In this section, the controller and the actuator are assumed to be separated. We consider transmission

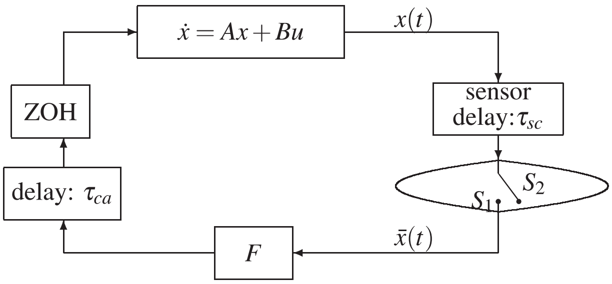

Fig. 2. An NCS with data packet dropout and transmission delays.

delays induced by the network: sensor-to-controller delay $\tau_{\rm sc}$ and controller-to-actuator delay $\tau_{\rm ca}$. In fact, if the feedback controller is static, these two delays can be lumped together. The NCS model with transmission delays and data packet dropout is shown in Fig. 2. The model consists of a continuous-time plant

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{17}$$

and a piecewise continuous controller (realized by a ZOH)

$$u(t) = -F\bar{x}(t - \tau_{\rm ca}), \quad t \in [t_k, t_{k-1}), \ k = 1, 2, \ldots. \tag{18}$$

The delay $\tau_{\rm ca}$ or $\tau_{\rm sc}$ may be less than or larger than one sampling period h. The network can be viewed as a switch, and the dynamics of the switch with transmission delays can be described as follows:

the NCS (17)–(18) with no packet dropout at time $t_k: \bar{x}(t) = x(t_k - \tau_{\rm sc})$;
the NCS (17)–(18) with one packet dropout at time $t_k: \bar{x}(t) = x(t_k - \tau_{\rm sc} - h)$;

$$\vdots$$

the NCS (17)–(18) with $d(k)$ packet dropout at time $t_k: \bar{x}(t) = x(t_k - \tau_{\rm sc} - d(k)h)$.

In this mode of switch, the quantity of dropped packet is accumulated from the latest time when $\bar{x}(t - \tau_{\rm ca})$ has been updated. Thus the closed-loop system with the effects of transmission delays and network packet loss is described as

$$\begin{aligned}\dot{x}(t) &= Ax(t) - BF\bar{x}(t - \tau_{\rm ca})\\ &= Ax(t) - BFx(t_k - \tau_{\rm ca} - \tau_{\rm sc} - d(k)h), \quad t \in [t_k, t_{k+1}).\end{aligned}$$

Let $\tau(t) = t - t_k + \tau_{\rm ca} + \tau_{\rm sc} + d(k)h$, then the system can be expressed as:

$$\dot{x}(t) = Ax(t) - BFx(t - \tau(t)), \quad t \in [t_k, t_{k-1}). \tag{19}$$

The quantity $d(k) \in Z^+$ may vary with time t and it is assumed that

$$0 \le d(k) \le d_k < \infty$$

for some positive integer d_k. Hence the delay function $\tau(t)$ satisfies

$$\begin{aligned}0 &\le \tau(t) = t - t_k + \tau_{\rm ca} + \tau_{\rm sc} + d(k)h\\ &\le (\bar{d} + 1)h + \tau_{\rm ca} + \tau_{\rm sc}, \quad t \in [t_k, t_{k+1}),\end{aligned}$$

where $\bar{d} = \max_k\{d_k\}$. Again, it is assumed that $(\bar{d} + 1)h + \tau_{\rm ca} + \tau_{\rm sc} \le \bar{\tau}$, where $\bar{\tau}$ is a positive scalar.

In this way, the NCS (17)–(18) with transmission delays and data packet dropout is modelled as a linear system with time-varying input delay (19), thus the existing theory of delay systems can be applied. It is noted that the delay function in system (19) does not satisfy the condition $\dot{\tau}(t) < 1$ as required in previous work (e.g. [8]).

The stabilization result on the NCS (17)–(18) with transmission delays and data packet dropout is a direct consequence of Theorem 1 and is briefly summarized as follows.

Theorem 2. Given an integer $\bar{\tau} > 0$, if there exist symmetric positive definite matrices $X, R_1, R_2 \in \mathbf{R}^{n\times n}$, a symmetric positive definite matrix $Q \in \mathbf{R}^{m\times m}$ and a matrix $Y \in \mathbf{R}^{m\times n}$ satisfying the following LMIs:

$$\begin{bmatrix} \frac{1}{\bar{\tau}}(XA + AX + Y^{\rm T}B^{\rm T} + BY) + 2X & BQ \\ QB^{\rm T} & -Q \end{bmatrix} < 0,$$

$$\begin{bmatrix} -X & Y^{\rm T}B^{\rm T} \\ BY & -R_2 \end{bmatrix} \le 0,$$

$$\begin{bmatrix} -X & XA^{\rm T} \\ AX & -R_1 \end{bmatrix} \le 0,$$

$$\begin{bmatrix} -Q & Y \\ Y^{\rm T} & -X \end{bmatrix} \le 0,$$

$$R_1 + R_2 - X \le 0,$$

then the NCS (17)–(18) is asymptotically stable with

$$F = YX^{-1}$$

for data packet dropout and transmission delays satisfying $0 \le (d_k + 1)h + \tau_{\rm ca} + \tau_{\rm sc} \le \bar{\tau}$.

Remark 3. The largest $\bar{\tau}$ preserving the stability of the NCS (17)–(18) can be determined by solving a quasi-convex optimization problem with the efficient LMI toolbox [2].

5. Stabilization of NCSs with Multiple-Packet Transmissions

In distributed NCSs, due to the wide location of sensors whose information length may surpass that of the network packet, multiple-packet transmission is necessary. In the multiple-packet transmission, plant/controller output is split into separate packets to be transmitted to the controller/actuator over different network channels and they may not arrive at the destination simultaneously.

An NCS with multiple-packet transmission and data packet dropout can be modelled as a linear system with multiple delays. For simplicity, we consider the case that the actuator and controller are combined into one node and there are no transmission delays.

Consider the NCS (1)–(2) with matrix B being of full column rank. For simplicity, we assume that the plant state is split into two parts $x(t) = \left[X_1^{\mathrm{T}}(t)\ X_2^{\mathrm{T}}(t)\right]^{\mathrm{T}}$ with $X_1(t)$ and $X_2(t)$ being transmitted over different network channels, where $X_1(t) = [x_1(t) \cdots x_r(t)]^{\mathrm{T}}$ and $X_2(t) = [x_{r+1}(t) \cdots x_n(t)]^{\mathrm{T}}, r < n$. Figure 3 illustrates the case where the plant state is transmitted in two packets.

Data packet dropout might happen over each network channel. When the switch is in position S_1, the outputs of S_1 and S_2 are held at the previous value and two packets are lost; When the switch is in position S_2, network packet containing $X_1(kh)$ is transmitted, whereas the output of switch S_2 is held at the previous value; Similar analysis can be done when the switch is in position S_3. Thus, the multiple-channel network is modelled as a switch, and the dynamics of the switch can be described as follows:

$$\bar{X}_1(t) = \bar{X}_1(t-h), \quad \bar{X}_2(t) = \bar{X}_2(t-h)$$
if the switch is in position S_1;
$$\bar{X}_1(t) = \bar{X}_1(t-h), \quad \bar{X}_2(t) = X_2(t_k)$$
if the switch is in position S_2;
$$\bar{X}_1(t) = X_1(t_k), \quad \bar{X}_2(t) = \bar{X}_2(t-h)$$
if the switch is in position S_3.

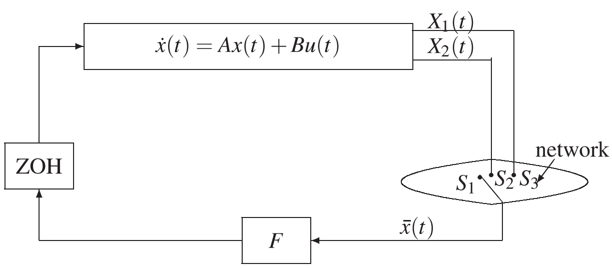

Fig. 3. An NCS with multiple packet transmission.

Analogous to the single-packet transmission case, we obtain that

$$\bar{x}(t) = \begin{bmatrix} \bar{X}_1(t) \\ \bar{X}_2(t) \end{bmatrix} = \begin{bmatrix} X_1(t_k - d_1(k)h) \\ X_2(t_k - d_2(k)h) \end{bmatrix} = \begin{bmatrix} X_1(t-\tau_1(t)) \\ X_2(t-\tau_2(t)) \end{bmatrix},$$

where $\tau_1(t) = t - t_k + d_1(k)h, \tau_2(t) = t - t_k + d_2(k)h$. Thus the closed-loop system with two packet transmission is modelled as

$$\begin{aligned} \dot{x}(t) = Ax(t) + BF\bar{x}(t) = Ax(t) + BFC_1 x(t-\tau_1(t)) \\ + BFC_2 x(t-\tau_2(t)), \end{aligned} \tag{20}$$

where

$$C_1 = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix}.$$

The quantity $d_i(k) \in Z^+ (i = 1, 2)$ may vary with time t and it is assumed that

$$0 \le d_i(k) \le d_{ik} < \infty$$

for some positive integers d_{ik}. Hence the delay $\tau_i(t)(i = 1, 2)$ satisfy the following inequality:

$$\begin{aligned} &0 \le \tau_i(t) = t - t_k + d_i(k)h \le (\bar{d}_i + 1)h, \\ &\quad t \in (t_k, t_{k+1}), \end{aligned}$$

where $\bar{d}_i = \max_k\{d_{ik}\}$. We also assume that $(\bar{d}_i + 1)$ $h \le \bar{\tau}_i$, where $\bar{\tau}_i$ are positive scalars.

As in the previous sections, the NCS (1)–(2) with data packet dropout in multiple-packet transmission can be modelled as a linear system with time-varying multiple input delays (20). We will apply the theory of delay systems to the analysis and design of the NCS (1)–(2). Due to the characteristics of the control problem under consideration and our modelling method, the original (state feedback) control problem for such a system can be viewed as an output feedback problem in our framework.

The following theorem establishes a sufficient condition on the stabilization of the NCS (1)–(2) with data packet dropout in multiple-packet transmission.

Theorem 3. Given integers $\bar{\tau}_1 > 0, \bar{\tau}_2 > 0$, if there exist symmetric positive definite matrices $P, U_1, U_2, U_3, U_4, U_5, U_6 \in \mathbf{R}^{n \times n}$, a matrix $M \in \mathbf{R}^{m \times m}$ and a matrix $N \in \mathbf{R}^{m \times n}$ satisfying

$$PB = BM, \tag{21}$$

and the following LMIs:

$$A^{\mathrm{T}} U_1 A \leq P, \quad A^{\mathrm{T}} U_4 A \leq P, \tag{22}$$

$$U_2 \leq P, \quad U_3 \leq P, \quad U_5 \leq P, \quad U_6 \leq P, \tag{23}$$

$$\begin{bmatrix} P & C_1^{\mathrm{T}} N^{\mathrm{T}} B^{\mathrm{T}} \\ BNC_1 & P \end{bmatrix} \geq 0, \tag{24}$$

$$\begin{bmatrix} P & C_2^{\mathrm{T}} N^{\mathrm{T}} B^{\mathrm{T}} \\ BNC_2 & P \end{bmatrix} \geq 0, \tag{25}$$

$$\begin{bmatrix} \Lambda & \bar{\tau}_1 BNC_1 & \bar{\tau}_1 BNC_1 & \bar{\tau}_1 BNC_1 & \bar{\tau}_2 BNC_2 & \bar{\tau}_2 BNC_2 & \bar{\tau}_2 BNC_2 \\ * & -\bar{\tau}_1 U_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\bar{\tau}_1 U_2 & 0 & 0 & 0 & 0 \\ * & * & * & -\bar{\tau}_1 U_3 & 0 & 0 & 0 \\ * & * & * & * & -\bar{\tau}_2 U_4 & 0 & 0 \\ * & * & * & * & * & -\bar{\tau}_2 U_5 & 0 \\ * & * & * & * & * & * & -\bar{\tau}_2 U_6 \end{bmatrix} < 0, \tag{26}$$

where $\Lambda = PA + A^{\mathrm{T}}P + BNC_1 + BNC_2 + C_1^{\mathrm{T}} N^{\mathrm{T}} B^{\mathrm{T}} + C_2^{\mathrm{T}} N^{\mathrm{T}} B^{\mathrm{T}} + 3(\bar{\tau}_1 + \bar{\tau}_2)P$, then the NCS (1)–(2) with B being of full column rank is asymptotically stable with

$$F = NM^{-1}$$

for data packet dropout satisfying $0 \leq d_{ik} \leq (\bar{\tau}_i/h) - 1$.

Proof. In each piecewise continuous interval of $\tau_i(t) (i = 1, 2)$, we apply the Newton–Leibniz formula to equation (20) and get

$$\begin{aligned} \dot{x}(t) = {} & (A + BFC_1 + BFC_2)x(t) \\ & - BFC_1 \int_{t-\tau_1(t)}^{t} (Ax(\theta) + BFC_1 x(\theta - \tau_1(\theta)) \\ & + BFC_2 x(\theta - \tau_2(\theta)))\, \mathrm{d}\theta \\ & - BFC_2 \int_{t-\tau_2(t)}^{t} (Ax(\theta) + BFC_1 x(\theta - \tau_1(\theta)) \\ & + BFC_2 x(\theta - \tau_2(\theta)))\, \mathrm{d}\theta \end{aligned} \tag{27}$$

with the initial condition $x(t_0 + \theta) = \phi(\theta)$ for $\theta \in \xi_{t_0}$, where ϕ is a continuous norm-bounded initial function and $\xi_{t_0} = \bigcup_{i=1}^{2} \xi_{t_0}^{i} = \bigcup_{i=1}^{2} \{t \in R : t = \eta - 2\tau_i(\eta) \leq t_0, \eta \geq t_0\}$. From [7], global uniform asymptotic stability of system (20) will be ensured by that of system (27). Thus we proceed to study the latter.

Consider the following Lyapunov function

$$V(x(t)) = x^{\mathrm{T}}(t) P x(t),$$

where P is a symmetric positive definite matrix. The time derivative of $V(x(t))$ along the trajectory of system (27) is given by

$$\begin{aligned} \dot{V}(x) = {} & x^{\mathrm{T}}(t)(P(A + BFC_1 + BFC_2) \\ & + (A + BFC_1 + BFC_2)^{\mathrm{T}} P)x(t) \\ & - 2x^{\mathrm{T}}(t) PBFC_1 \int_{t-\tau_1(t)}^{t} (Ax(\theta) \\ & + BFC_1 x(\theta - \tau_1(\theta)) + BFC_2 x(\theta - \tau_1(\theta)))\, \mathrm{d}\theta \\ & - 2x^{\mathrm{T}}(t) PBFC_2 \int_{t-\tau_2(t)}^{t} (Ax(\theta) \\ & + BFC_1 x(\theta - \tau_2(\theta)) + BFC_2 x(\theta - \tau_2(\theta)))\, \mathrm{d}\theta. \end{aligned} \tag{28}$$

Let $R_i = U_i^{-1}, i = 1, \ldots, 6$. From (22)–(23), we obtain

$$A^{\mathrm{T}} R_1^{-1} A \leq P, \quad A^{\mathrm{T}} R_4^{-1} A \leq P, \tag{29}$$

$$R_2^{-1} \leq P, \quad R_3^{-1} \leq P, \quad R_5^{-1} \leq P, \quad R_6^{-1} \leq P. \tag{30}$$

Because B is of full column rank, it follows from (21) that M is also of full rank, and thus invertible. Using this fact and Schur complement, we can deduce from (24)–(25) and (30) that

$$\begin{aligned} C_1^{\mathrm{T}} F^{\mathrm{T}} B^{\mathrm{T}} R_2^{-1} BFC_1 & \leq C_1^{\mathrm{T}} F^{\mathrm{T}} B^{\mathrm{T}} PP^{-1} PBFC_1 \\ & = C_1^{\mathrm{T}} N^{\mathrm{T}} B^{\mathrm{T}} P^{-1} BNC_1 \leq P, \end{aligned} \tag{31}$$

$$\begin{aligned} C_2^{\mathrm{T}} F^{\mathrm{T}} B^{\mathrm{T}} R_3^{-1} BFC_2 & \leq C_2^{\mathrm{T}} F^{\mathrm{T}} B^{\mathrm{T}} PP^{-1} PBFC_2 \\ & = C_2^{\mathrm{T}} N^{\mathrm{T}} B^{\mathrm{T}} P^{-1} BNC_2 \leq P, \end{aligned} \tag{32}$$

$$\begin{aligned} C_1^{\mathrm{T}} F^{\mathrm{T}} B^{\mathrm{T}} R_5^{-1} BFC_1 & \leq C_1^{\mathrm{T}} F^{\mathrm{T}} B^{\mathrm{T}} PP^{-1} PBFC_1 \\ & = C_1^{\mathrm{T}} N^{\mathrm{T}} B^{\mathrm{T}} P^{-1} BNC_1 \leq P, \end{aligned} \tag{33}$$

$$\begin{aligned} C_2^{\mathrm{T}} F^{\mathrm{T}} B^{\mathrm{T}} R_6^{-1} BFC_2 & \leq C_2^{\mathrm{T}} F^{\mathrm{T}} B^{\mathrm{T}} PP^{-1} PBFC_2 \\ & = C_2^{\mathrm{T}} N^{\mathrm{T}} B^{\mathrm{T}} P^{-1} BNC_2 \leq P, \end{aligned} \tag{34}$$

where $N = MF$.

To employ the Razumikhin-type theorem [7], we evaluate $\dot{V}(x(t))$ for the case

$$V(x(\eta)) < \delta V(x(t)), \quad t - 2\tau \le \eta \le t, \tag{35}$$

with $\delta > 1$. By Lemma 1, it follows from (29)–(35) that

$$\begin{aligned}
&-2x^{\mathrm{T}} PBFC_1 \int_{t-\tau_1(t)}^{t} Ax(\theta)\,\mathrm{d}\theta \\
&\quad \le \int_{t-\tau_1(t)}^{t} x^{\mathrm{T}}(\theta) A^{\mathrm{T}} R_1^{-1} Ax(\theta)\,\mathrm{d}\theta \\
&\qquad + \tau_1(t) x^{\mathrm{T}}(t) PBFC_1 R_1 C_1^{\mathrm{T}} F^{\mathrm{T}} B^{\mathrm{T}} Px(t) \\
&\quad \le \int_{t-\tau_1(t)}^{t} x^{\mathrm{T}}(\theta) Px(\theta)\,\mathrm{d}\theta \\
&\qquad + \tau_1(t) x^{\mathrm{T}}(t) PBFC_1 R_1 C_1^{\mathrm{T}} F^{\mathrm{T}} B^{\mathrm{T}} Px(t) \\
&\quad \le x^{\mathrm{T}}(t)\tau_1(t)(\delta P + PBFC_1 R_1 C_1^{\mathrm{T}} F^{\mathrm{T}} B^{\mathrm{T}} P)x(t), \\
&-2x^{\mathrm{T}} PBFC_1 \int_{t-\tau_1(t)}^{t} BFC_1 x(\theta - \tau_1(\theta))\,\mathrm{d}\theta \\
&\quad \le x^{\mathrm{T}}(t)\tau_1(t)(\delta P + PBFC_1 R_2 C_1^{\mathrm{T}} F^{\mathrm{T}} B^{\mathrm{T}} P)x(t), \\
&-2x^{\mathrm{T}} PBFC_1 \int_{t-\tau_1(t)}^{t} BFC_2 x(\theta - \tau_2(\theta))\,\mathrm{d}\theta \\
&\quad \le x^{\mathrm{T}}(t)\tau_1(t)(\delta P + PBFC_1 R_3 C_1^{\mathrm{T}} F^{\mathrm{T}} B^{\mathrm{T}} P)x(t), \\
&-2x^{\mathrm{T}} PBFC_2 \int_{t-\tau_2(t)}^{t} Ax(\theta)\,\mathrm{d}\theta \\
&\quad \le x^{\mathrm{T}}(t)\tau_2(t)(\delta P + PBFC_2 R_4 C_2^{\mathrm{T}} F^{\mathrm{T}} B^{\mathrm{T}} P)x(t), \\
&-2x^{\mathrm{T}} PBFC_2 \int_{t-\tau_2(t)}^{t} BFC_1 x(\theta - \tau_1(\theta))\,\mathrm{d}\theta \\
&\quad \le x^{\mathrm{T}}(t)\tau_2(t)(\delta P + PBFC_2 R_5 C_2^{\mathrm{T}} F^{\mathrm{T}} B^{\mathrm{T}} P)x(t), \\
&-2x^{\mathrm{T}} PBFC_2 \int_{t-\tau_2(t)}^{t} BFC_2 x(\theta - \tau_2(\theta))\,\mathrm{d}\theta \\
&\quad \le x^{\mathrm{T}}(t)\tau_2(t)(\delta P + PBFC_2 R_6 C_2^{\mathrm{T}} F^{\mathrm{T}} B^{\mathrm{T}} P)x(t).
\end{aligned}$$

Then, we obtain that

$$\begin{aligned}
&\dot{V}(x(t)) \\
&\quad \le x^{\mathrm{T}}(t)(P(A + BFC_1 + BFC_2) \\
&\qquad + (A + BFC_1 + BFC_2)^{\mathrm{T}} P \\
&\qquad + \tau_1(t)(3\delta P + PBFC_1(R_1 + R_2 + R_3)C_1^{\mathrm{T}} F^{\mathrm{T}} B^{\mathrm{T}} P) \\
&\qquad + \tau_2(t)(3\delta P + PBFC_2(R_4 + R_5 + R_6)C_2^{\mathrm{T}} F^{\mathrm{T}} B^{\mathrm{T}} P)) \\
&\qquad \times x(t) \\
&\quad = x^{\mathrm{T}}(t)(PA + BNC_1 + BNC_2 + A^{\mathrm{T}} P + C_1^{\mathrm{T}} N^{\mathrm{T}} B^{\mathrm{T}} \\
&\qquad + C_2^{\mathrm{T}} N^{\mathrm{T}} B^{\mathrm{T}} + \tau_1(t)(3\delta P + BNC_1(R_1 + R_2 + R_3) \\
&\qquad \times C_1^{\mathrm{T}} N^{\mathrm{T}} B^{\mathrm{T}}) + \tau_2(t)(3\delta P + BNC_2(R_4 + R_5 + R_6) \\
&\qquad \times C_2^{\mathrm{T}} N^{\mathrm{T}} B^{\mathrm{T}}))x(t).
\end{aligned}$$

Hence, $\dot{V}(x(t)) < 0$ for $V(x(\theta)) < \delta V(x(t))(t - 2\tau \le \theta \le t)$ if

$$\begin{aligned}
&PA + BNC_1 + BNC_2 + A^{\mathrm{T}} P + C_1^{\mathrm{T}} N^{\mathrm{T}} B^{\mathrm{T}} + C_2^{\mathrm{T}} N^{\mathrm{T}} B^{\mathrm{T}} \\
&\quad + \tau_1(t)(3\delta P + BNC_1(R_1 + R_2 + R_3)C_1^{\mathrm{T}} N^{\mathrm{T}} B^{\mathrm{T}}) \\
&\quad + \tau_2(t)(3\delta P + BNC_2(R_4 + R_5 + R_6)C_2^{\mathrm{T}} N^{\mathrm{T}} B^{\mathrm{T}}) < 0.
\end{aligned} \tag{36}$$

Using Schur complement, (26) is equivalent to

$$\begin{aligned}
&PA + BNC_1 + BNC_2 + A^{\mathrm{T}} P + C_1^{\mathrm{T}} N^{\mathrm{T}} B^{\mathrm{T}} + C_2^{\mathrm{T}} N^{\mathrm{T}} B^{\mathrm{T}} \\
&\quad + \bar{\tau}_1(3P + BNC_1(R_1 + R_2 + R_3)C_1^{\mathrm{T}} N^{\mathrm{T}} B^{\mathrm{T}}) \\
&\quad + \bar{\tau}_2(3P + BNC_2(R_4 + R_5 + R_6)C_2^{\mathrm{T}} N^{\mathrm{T}} B^{\mathrm{T}}) < 0.
\end{aligned} \tag{37}$$

By the continuity of (36) in δ, (26) guarantees that there exists a scalar $\delta > 1$ sufficient small such that (36) holds for $\tau_i(t) \le \bar{\tau}_i, i = 1, 2$. This completes the proof. □

For the NCS (1)–(2) in multiple-packet transmission, by Theorem 3, the feedback controller can be obtained in terms of LMIs. For admissible bound on the amount of data packet loss in each network channel, if $\bar{\tau}_1$ is set to be fixed, then $\bar{\tau}_2$ can be maximized via the quasi-convex optimization algorithm with the efficient LMI toolbox [2]; or vice versa.

Remark 4. It is noted that two matrices C_1 and C_2 with special structures have been introduced into the closed-loop system and this introduces additional complexity in feedback controller design. To tackle this problem, some bounds for the cross-product terms have been introduced and, admittedly, this introduces some conservatism.

Remark 5. Theorem 3 can be easily extended to NCSs with transmission delays and data packet dropout in multiple-packet transmission and to NCSs with separate controller and actuator. Moreover, though we present our results for two-packet transmission case simply for notational simplicity, it should be clear that our results and methods are not restricted to the two-packet transmission case.

Remark 6. In a very recent paper [6], a sampled-data system is modelled as a linear system with time-varying delays without any restriction on the derivative of the delay function. However, the method presented there must first fix a tuning parameter or use an iterative algorithm to obtain the stabilizing controller in terms of LMIs. Moreover, the results in [6] only concern the case of a system with a single delay function, and can not be readily and directly applied to a system with multiple delay functions as in (20) (which corresponds to the multiple-packet transmission case).

6. Numerical Examples

Example 1. Consider the state-space plant transmitted in single packet

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -0.01 \\ 1 & 0.02 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.4 \\ 0.1 \end{bmatrix} u. \tag{38}$$

The feedback controller takes the form $u = -F\bar{x}(t)$, with F to be designed.

When the state information from the plant to the controller is transmitted through a network channel, system (38) can be written in the form of (3). By solving the quasi-convex optimization problem (16) (based on Theorem 1) using the LMI toolbox [2], we obtain the feedback gain $F = [-3.8625 \quad -3.9211]$ and the admissible bound of $\bar{\tau}$ to be 0.5950. Therefore, if the sampling period $h = 0.19$ s and the transmission delays can be neglected, the closed-loop system is still stable even in the case of two data packets dropped in every three packets. From the relationship between the dropped packet and $\bar{\tau}$, we can see that, with the increase of the sampling period, the quantity of data packet dropout between two transmission instants has to be decreased in order to guarantee the stability of the NCS.

Example 2. Consider the state-space plant transmitted in two packets.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -0.8 & 0.1 \\ 0.2 & 0.05 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u. \tag{39}$$

The feedback controller takes the form $u = -F\bar{x}(t)$, with F to be designed.

In this example, we consider stabilization of the system with data packet dropout. It is assumed that $X_1 = x_1,\ X_2 = x_2$. From Section 5 we obtain

$$C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

For simplicity, we assume that $\bar{\tau}_1 = \bar{\tau}_2$. The admissible upper bound is found to be $\bar{\tau}_i = 0.3102$ $(i = 1, 2)$ using the quasi-convex optimization algorithm based on Theorem 3. Also, we can obtain the feedback gain $F = [0.0756 \quad -4.1729]$. This shows that for given sampling period h, if data packet dropout in each network channel satisfies $0 \leq d_{ik} \leq (\bar{\tau}_i/h) - 1$, the NCS (39) is stable.

For the specific case $\tau_1(t) = \tau_2(t)$, Remark 1 can be applied. Solving the quasi-convex optimization problem (16), the upper bound of $\tau_i(t)(i = 1, 2)$ and the feedback gain are found to be 0.5295 and $F = [-0.1049\ -1.2200]$. This shows that, in this specific case, Theorem 3 is a bit more conservative than Theorem 1 because some bounds were introduced in order to deal with the cross-product terms.

These two examples illustrate that with the increase of the sampling period, the data packet dropout has to be decreased, that is, the NCS will be sensitive to data packet dropout in the case of slow sampling. Though it is natural to sample fast to approximate the real continuous-time system in order to be robust to data packet dropout, this will increase the load of the network and thus deteriorate its performance. Hence, it is an interesting and important research topic to study the relationship between sampling period, performance and data packet dropout of NCSs.

7. Conclusions

We have investigated the stabilization problem for a class of networked control systems with data packet dropout and transmission delays induced by network channels. The admissible upper bound of data packet dropout and transmission delays can be obtained via solving a quasi-convex optimization problem with the efficient LMI toolbox. The feedback controllers can be constructed in terms of LMIs. For NCSs in multiple-packet transmission, similar stabilization results have been established, where single delay has been extended to multiple delays, and state feedback problem has been extended to output feedback problem. Illustrative examples on both single-packet transmission and multiple-packet transmission have been worked out to demonstrate the effectiveness of the proposed approach. The obtained results in this paper can be applied to NCSs without packet loss to save network bandwidth or to find the maximum allowable delay between state update. This is of practical importance in engineering applications.

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