Transmission Lines

> The difference between electricity and electronics is the difference between a toaster and a television set.
>
> Isaac Asimov (1920 - 1992)

OBJECTIVES

To investigate some properties of electromagnetic waves in transmission lines.

THEORY

Electromagnetic radiation can be contained and guided within structures called transmission lines. There are many examples of transmission lines in common use. Twin lead, often used as television antenna cable, consists of two parallel conductors separated by a plastic spacer. A better system for transmitting radio frequency signals (up to about 10 GHz) is the coaxial cable. The coaxial cable has cylindrical symmetry: a center conductor is surrounded by a "shield" conductor which is usually at ground potential. The shield helps to reduce noise pickup in the signal line. The BNC connectors on our oscilloscopes also have this coaxial geometry. At microwave frequencies, 3 GHz ~ 300 GHz, radiation can be transmitted in hollow metal pipes called wave guides. Wave guides usually have a rectangular cross section whose sides are approximately one half wavelength (~1 mm ~ 10 cm) in length. In the optical domain, waves can be transmitted by "optical fiber". Optical fiber is a transparent material which has a higher index of refraction than the surroundings so that propagating waves are totally internally reflected.

In addition to their technological importance, there is a wealth of interesting physics which can be studied in transmission lines. Propagating waves in transmission lines will reflect from abrupt discontinuities (i.e. changes which occur on length scales shorter than a wavelength) just as optical waves partially reflect off boundaries between two materials of different index of refraction. If two boundaries are present, standing waves can occur, as on the vibrating string you studied earlier. Alternatively, we can view the transmission line as an electrical circuit problem. Then we would want to know the current and voltage at each point in the line. The wave and circuit approaches are both useful, as you will see in carrying out this experiment.

The electrical properties of a transmission line can be derived directly from Maxwell's equations, but the mathematics is very complicated. We will, instead, model the line as though it were an electrical circuit. To do this, we imagine dividing it into sections of length Δx characterized by a combination of capacitance, inductance and resistance per unit length, as shown in Fig. 1. The capacitance is due to the electric fields between the two conductors, the

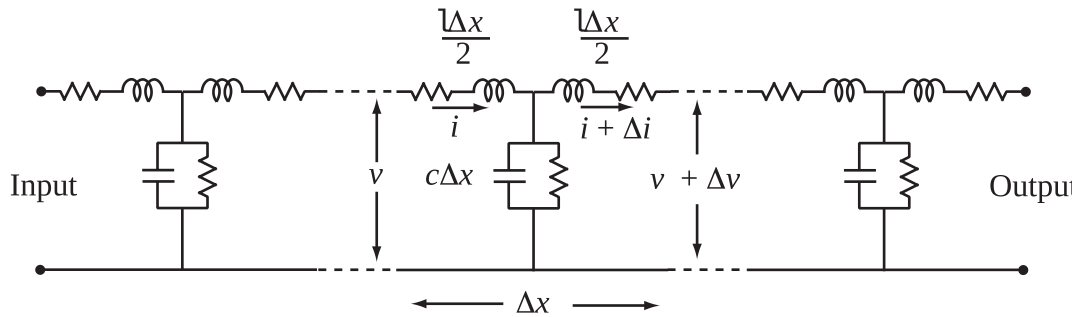

Fig. 1 Equivalent circuit of a small length of transmission line. As indicated, the model for the whole transmission line consists of a large number of these small sections. The quantities ℓ and c are given per unit length of the line.

inductance describes the effect of the magnetic fields, and the resistance models the energy losses in the metal wires and insulating materials. For technically useful transmission lines the energy losses are small, so we will henceforth neglect the resistance terms.

In keeping with our circuit model, we will analyze the response of the line by applying Kirchoff's laws to each segment Δx. Consider one section, as marked in the figure, for which the input is $v(x)$, $i(x)$ and the output is $v(x)+\Delta v$, $i(x)+\Delta i$. The voltage equation for the loop through the two inductors is

$$v - \frac{1}{2}\ell\Delta x\frac{di}{dt} - \frac{1}{2}\ell\Delta x\frac{d}{dt}(i + \Delta i) - (v + \Delta v) = 0 \tag{1}$$

Dividing through by Δx and taking the limit $\Delta x \to 0$, we obtain

$$\frac{\partial i}{\partial t} = -\frac{1}{\ell}\frac{\partial v}{\partial x} \tag{2}$$

For the loop through one inductor and the capacitor, we get

$$v - \frac{1}{2}\ell\Delta x\frac{di}{dt} - \frac{q}{c\Delta x} = 0 \tag{3}$$

Differentiating with respect to time and using the fact that $dq/dt = i - (i + \Delta i)$ yields

$$\frac{dv}{dt} - \frac{1}{2}\ell\Delta x\frac{d^2 i}{dt^2} + \frac{\Delta i}{c\Delta x} = 0 \tag{4}$$

Taking the limit gives

$$\frac{\partial v}{\partial t} = -\frac{1}{c}\frac{\partial i}{\partial x} \tag{5}$$

Differentiating Eq. 2 with respect to x and Eq. 5 with respect to t gives two equations which can be combined to give a wave equation

$$\frac{\partial^2 v}{\partial x^2} - \ell c \frac{\partial^2 v}{\partial t^2} = 0 \tag{6}$$

The general solution to Eq. 6 is

$$v(x,t) = f(x - st) + g(x + st) \tag{7}$$

with propagation speed $s = (\ell c)^{-1/2}$. By substituting back into Eq. 2 and 5, one can show that the corresponding solution for the current is

$$i(x,t) = \frac{1}{Z_c} f(x - st) - \frac{1}{Z_c} g(x + st) \tag{8}$$

where $Z_C = (\ell/c)^{1/2}$ is called the characteristic impedance of the cable. Evidently the transmission line can support waves of current and voltage traveling in either direction. The quantity Z_C is an impedance because it tells us how much current flows for a given applied voltage. For our lossless line Z_C is real, that is a pure resistance. If we had included energy losses in the conductors and dielectric it would have an imaginary part and the wave equation would be much more complicated.

To proceed further we need to specify some particular situation, such as the set-up of Fig. 2. The signal generator starts a pulse down a transmission line of length d which is terminated with a load Z_L. When the pulse reaches the far end we expect it to reflect and travel back toward the source. We can find the reflected signal using the general solutions for $v(x,t)$ and $i(x,t)$ at the far end of the transmission line. Applying Kirchoff's Laws to the loop at the output end we obtain

$$v(d,t) - i(d,t)Z_L = 0 \tag{9}$$

where $v(d,t)$ and $i(d,t)$ are given by Eq. 7 and 8. Substituting and solving for $g(d + st)$ we obtain

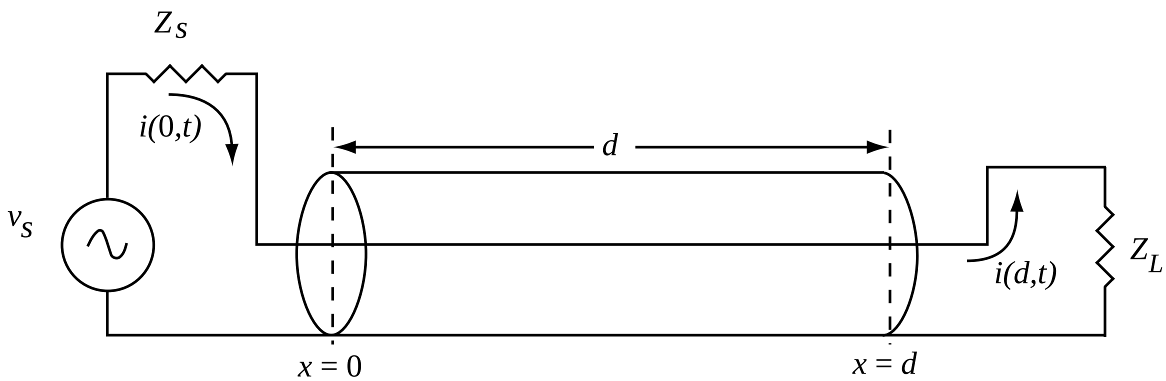

Fig. 2 Finite length of transmission line driven by a voltage source and terminated at $x = d$ with the load impedance Z_L.

$$g(d + st) = rf(d - st) \tag{10}$$

with reflection coefficient r given by

$$r = \frac{Z_L - Z_C}{Z_L + Z_C} \tag{11}$$

The interpretation of this result is that the reflected wave is a copy of the incident pulse, but "flipped over" to travel in the opposite direction and with a different amplitude.

Not surprisingly, the amount of reflection depends on the terminating impedance, reaching total reflection ($r = \pm 1$) when the load impedance is infinite or zero. These limits can be understood physically by realizing that the current or voltage, respectively, of the incident and reflected pulses must exactly cancel at the end. The special case $Z_L = Z_c$ is also of interest, since then $r = 0$ and there is no reflection at all. This situation is called a "matched termination" or just a "termination", and is often a desirable way to minimize interfering reflections in practical situations.

If we now imagine that our signal generator produces sine waves, we can set up standing waves on the transmission line, much as we did on the string. Here, however, we can change the termination, and will in fact evaluate the voltage at the input, $v(0,t)$, for both infinite and zero load impedances. The general harmonic solution is

$$v(x,t) = A\mathrm{e}^{i(\omega t - kx)} + B\mathrm{e}^{i(\omega t + kx)} \tag{12}$$

where k is the propagation constant, $2\pi/\lambda$. From the wave equation we get the dispersion relation for the cable:

$$\omega = k(\ell c)^{-1/2} \tag{13}$$

Inserting $v(x,t)$ from Eq. 12 into the loop equation 9 we eventually find $B = rA\exp(-2ikd)$, leading to the specific solution

$$v(0,t) = A\left(1 + r\mathrm{e}^{-2ikd}\right)\mathrm{e}^{i\omega t} \tag{14}$$

When Z_L is infinite, $r = +1$ and therefore $v(0,t) = 0$ whenever $\exp(-2ikd) = -1$. This means that we will measure zero voltage at the input when d is an odd multiple of a quarter-wavelength, that is when $d = \lambda/4$, $3\lambda/4$.... Similarly, when Z_L is zero, $r = -1$, and there will be zero voltage at the input when d is a multiple of a half wavelength, $d = \lambda/2$, λ, ...

The last topic we will consider is the power dissipation in a terminating resistor, R_L. When there is a time-dependent voltage v across a resistor R_L, the power dissipated in the resistor is just

$$P = \frac{\langle v \rangle^2}{R_L} \tag{15}$$

where the brackets denote an RMS average. For the harmonically driven transmission line,

$$v = v(d,t) = A(1+r)\mathrm{e}^{-ikd}\mathrm{e}^{i\omega t} \tag{16}$$

which we can compute from our general solutions once we have an expression for A. This is found by solving the loop equation for the input loop of Fig. 2. The algebra is slightly messy, so we only quote the result when $Z_s = Z_C$, which is a good approximation for our apparatus:

$$A = \frac{v_s}{2} \tag{17}$$

From this we get

$$P = \frac{v_s^2}{2}\frac{R_L}{\left(R_L + Z_C\right)^2} \tag{18}$$

This function approaches zero as R_L approaches zero or infinity and has a broad maximum at $R_L = Z_c$. So, to achieve maximum power transfer from source to load, the load (and source) must be matched to the characteristic impedance of the line.

EXPERIMENTAL PROCEDURE

The basic experimental set-up is shown in Fig. 3. The transmission line under study is a piece of 72Ω coaxial cable 4.3 meters long. Two signal sources are available, one for pulses and one for sine waves. Both have output impedances of about 70Ω, so there should be very little reflection at the input end. The measuring device is an oscilloscope, schematically shown as a voltmeter and input impedance Z_m (which includes a contribution from the connecting cable). All components are fitted with coaxial BNC connectors. In addition to the obvious complication of dealing with Z_S and Z_m, the experimenter needs to be aware of the fact that both impedances and the source voltage v_S may be functions of frequency.

1. Pulse measurements

The first set of data will use pulse signals to determine the speed of propagation and the reflection coefficient. When supplied with 10 V DC from the low-voltage power supply the pulser unit produces very short voltage pulses at a repetition rate of a few kilohertz.

Using a short cable, connect the pulser to a BNC tee at the scope input, and connect the long cable to the other arm of the tee. If the pulser is running you can now display the transmitted and reflected pulses on the scope screen. Be sure the scope is set for its full 60 MHz bandwidth so you have maximum time resolution. From the pulse transit time, estimate the propagation velocity for the cable.

A number of resistors on BNC connectors are mounted on the component board. Using the relative pulse heights, determine the reflection coefficient, including the sign of r, for each

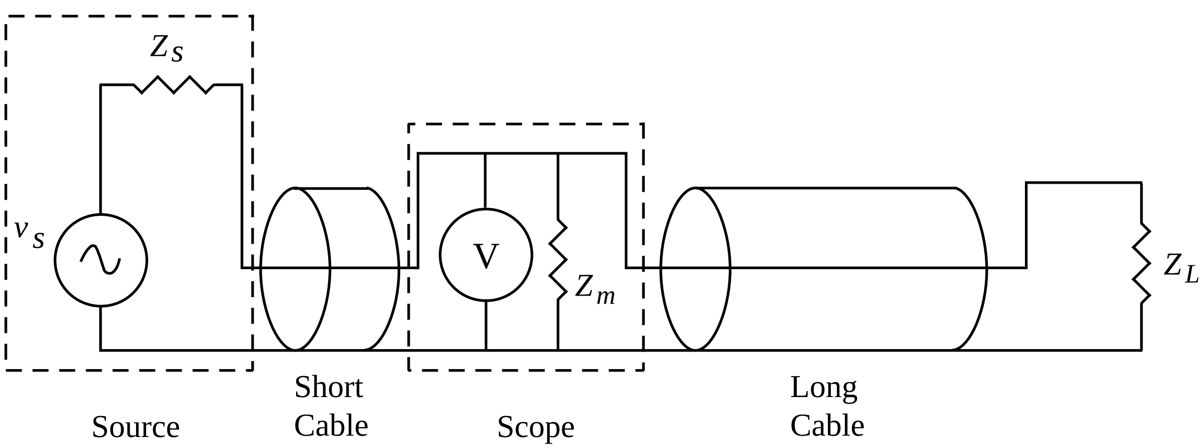

Fig. 3 Schematic of the experimental setup.

available termination. Are the reflection coefficients consistent with Eq. 11 and the claimed Z_c = 72Ω?

Some pieces of 50Ω cable are also available in the lab. If you wish, you may connect one of these to the end of your 72Ω cable to see the effects of the impedance mismatch. How do you interpret the sequence of pulses you see?

2. Standing wave measurements

Sine wave signals can be used to check features of the harmonic solutions and to determine the wavelength on the cable. From the equivalent circuit of Fig. 3, we expect that connecting the cable will affect the voltage measured on the scope. It would be simple to calculate this effect if we knew the impedances, but we don't. Also, Z_m is not necessarily much larger than Z_s, and everything varies with frequency. To minimize these complications we will not consider a general case, but rather try to learn as much as we can from the frequencies where $v(0,t)$ is zero.

Start by setting the signal generator modulation switch to external (we won't use the modulation feature in this experiment), and the amplitude knob for maximum output on the LO range. With these settings the generator produces a fairly pure sine wave and has an output impedance of very nearly 72Ω. Connect the generator to the tee at the scope input with a short piece of coax. Note that the amplitude of the displayed sine wave depends on frequency even when the long cable is not connected.

To test Eq. 14 locate the frequencies at which $v(0,t)$ is approximately zero when the load impedance Z_L is zero ohms or infinite ohms (disconnect the termination entirely). Are your results qualitatively consistent with Eq. 14? That is, are the zeros for either termination evenly spaced, with the zeros for the other termination falling between? Explain these results physically, including a sketch showing $v(x)$ along the cable for representative frequencies where you found $v(x{=}0) = 0$.

For a more quantitative test, locate as accurately as possible the frequencies at which $v(0,t)$ goes to zero for the open and short-circuit terminations. Use these data and the known cable length, d = 4.3 m, to deduce the wavelength in the cable and hence the wavevector k corresponding to each frequency. From our calculations, we expect k to depend linearly on frequency according to $k = \omega\,(\ell c)^{1/2}$. Prepare a graph to find out if this relation holds, and to determine the value of $(\ell c)^{1/2}$. Is the value consistent with your pulse measurement of the velocity?

3. Power transfer

To measure the power dissipation in the load we need to move to the other end of the transmission line. Connect one end of the long cable directly to the RF generator and the other end to a tee on one of the terminating resistors. Use the short cable to go from the termination to the scope. There is no obvious way to compensate for Z_m, so the measurement is best done at a fairly low frequency, say 1 or 2 MHz, where the input impedance of the scope will be much larger than the terminating impedance.

Measure the voltage across each of the BNC terminating resistors on the component board and calculate the power dissipation from Eq. 15. At these low frequencies you can get a good estimate of the source voltage v_s by disconnecting the load resistor and letting the function generator drive the scope directly. Check to see if the variation in power with load follows Eq. 18 when you use the known values of R_L and Z_c.

REPORT

Your report should include answers to the questions in the text, the data and plots requested and a physical explanation of your results where appropriate.